GENERICS FOR COMPUTABLE MATHIAS FORCING

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Abstract. We study the complexity of generic reals for computable Mathias forcing in the context of computability theory. The n-generics and weak n-generics form a strict hierarchy under Turing reducibility, as in the case of Cohen forcing. We analyze the complexity of the Mathias forcing relation, and show that if G is any n-generic with $n \geq 2$ then it satisfies the jump property $G^{(n-1)} \equiv_T G' \oplus \emptyset^{(n)}$. We prove that every such G has generalized high Turing degree, and so cannot have even Cohen 1-generic degree. On the other hand, we show that every Mathias n-generic real computes a Cohen n-generic real.

1. Introduction

Forcing has been a central technique in computability theory since it was introduced (in the form we now call Cohen forcing) by Kleene and Post to exhibit a degree strictly between $\mathbf{0}$ and $\mathbf{0}'$. The study of the algorithmic properties of Cohen generic reals, and of the structure of their degrees, has long been a rich source of problems and results. In the present article, we propose to undertake a similar investigation of generic reals for (computable) Mathias forcing, and present some of our initial results in this direction.

The Mathias forcing partial order is defined as follows.

Definition 1.1. A *condition* is a pair (D, E) where D is a finite subset of ω, E is an infinite such subset, and $\max D < \min E$. A condition (D^*, E^*) *extends* (D, E) if $D \subseteq D^* \subseteq D \cup E$ and $E^* \subseteq E$.

Intuitively, the finite set D represents a commitment of information, positive and negative, about a set to be constructed, and E represent a commitment of negative information alone. Thus, for instance, the condition $(\{5, 6\}, \{9, 11, 13, \ldots\})$ commits our set to contain 5 and 6, but no other even numbers, or odd numbers less than 9.

Mathias forcing gained prominence in set theory in the article [10], for whose author it has come to be named. In a restricted form, it was used even earlier by Soare [15], to build an infinite set with no subset of strictly higher Turing degree. In computability theory, it has subsequently become a prominent tool for constructing infinite homogeneous sets for computable colorings of pairs of integers, as in Seetapun and Slaman [12], Cholak, Jockusch, and Slaman [3], and Dzhafarov

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and Jockusch [6]. It has also found applications in algorithmic randomness, in Binns, Kjos-Hanssen, Lerman, and Solomon [1].

Our interest below will be in *computable* Mathias forcing, where the conditions are pairs (D, E) such that E is an infinite computable set. Other effective variants have been studied in the literature, such as when E is low or low$_2$, and many of our techniques below can be appropriately modified to obtain analogous results for these versions. We shall show below that a number of results for Cohen genericity hold also for Mathias genericity, but that a number of important ones do not. The main source of distinction, as we shall see, is that neither the partial order, nor the forcing relation, is computable in this setting, so many usual techniques do not carry over.

The article is organized as follows. In Section 2, we lay out a framework for working with Mathias forcing in computability theory, and use it to prove some basic results about Mathias generics, in addition to listing several previously known ones. In Section 3 we define and study the Mathias forcing relation, and characterize the complexity of forcing arithmetical formulas according to their quantifier depth. Section 4 proves a number of results concerning the Turing degrees of Mathias generic reals, including that they are all generalized high. In Section 5, we then prove that, level by level, Mathias generic reals compute Cohen generic reals.

We refer the reader to Soare [14] for general background on computability theory.

2. Definitions and basic results

We assume familiarity with the basics of forcing in arithmetic, as presented, e.g., in Shore [13, Chapter 3]. Throughout, *Cohen forcing* will refer to the space of finite binary strings, $2^{<\omega}$, partially ordered by the usual extension relation, $\preceq$. For further background on Cohen forcing specifically, see [4, Section 1.24].

Formalizing Definition 1.1 in the setting of computability theory requires some care. A slightly different presentation is given in [1, Section 6], over which ours has the benefit of reducing the complexity of the set of conditions from Σ^0_3 to Π^0_2.

Definition 2.1.

(1) A *(computable Mathias) pre-condition* is a pair (D, E) where D is a finite set, E is a computable set, and $\max D < \min E$.
(2) A *(computable Mathias) condition* is a pre-condition (D, E), such that E is infinite.
(3) A pre-condition (D^*, E^*) *extends* a pre-condition (D, E), written $(D^*, E^*) \leq (D, E)$, if $D \subseteq D^* \subseteq D \cup E$ and $E^* \subseteq E$.
(4) A set A *satisfies* a pre-condition (D, E) if $D \subseteq A \subseteq D \cup E$.

We say $(D^*, E^*) \leq (D, E)$ is a *finite extension* if $E - E^*$ is finite.

By an *index* for a pre-condition (D, E) we shall mean a pair (d, e) such that d is the canonical index of D and $E = \{x : \Phi_e(x) \downarrow = 1\}$. We adopt the convention that for all x, if $\Phi_e(x) \downarrow$ then $\Phi_e(y) \downarrow \in \{0, 1\}$ for all $y \leq x$. Thus, if E is infinite, i.e., if (D, E) is a condition, then Φ_e is total. Of course, if E is finite then Φ_e may only be partial, in which case it will be defined on a proper initial segment of ω.

The definition makes the set of all indices Π^0_1. However, we can pass to a computable subset containing an index for every pre-condition. Namely, define a strictly increasing computable function g by

$$\Phi_{g(d,e)}(x) = \begin{cases} 0 & \text{if } x \leq \max D_d, \\ \Phi_e(x) & \text{otherwise.} \end{cases}$$

Then the set of pairs of the form $(d, g(d, e))$ is computable, and each is an index for a pre-condition. Moreover, if (d, e) is such an index as well, then it and $(d, g(d, e))$ are indices for the same pre-condition. Though we shall not be explicit about it, all our references to pre-conditions in the sequel should formally be regarded as references to indices from this set. Further, for notational convenience, we shall sometimes identify a pre-condition (D, E) with its index, thereby treating D and E as numbers.

Note that whether one pre-condition extends another is a Π^0_2 question. By our convention about partial computable functions, the same question for conditions is readily seen to be Π^0_1.

In what follows, a Σ^0_n *set of conditions* refers to a Σ^0_n-definable set of pre-conditions, each of which is a condition. (This is not the same as the set of all conditions satisfying a given Σ^0_n definition, as discussed further in the next section. See also Remark 2.3.) As usual, we call a set of conditions *dense* if it contains an extension of every condition.

Definition 2.2. Fix $n \in \omega$.

(1) A set A *meets* a set $\mathcal{C}$ of conditions if it satisfies some member of $\mathcal{C}$.
(2) A set A *avoids* a set $\mathcal{C}$ of conditions if it meets the set of conditions having no extension in $\mathcal{C}$.
(3) A set G is *Mathias n-generic* if it meets or avoids every Σ^0_n set of conditions.
(4) A set G is *weakly Mathias n-generic* if it meets every dense such set.

We call a set *generic* if it is n-generic for all n. We call a degree *n-generic*, or *generic*, if it contains an n-generic, or generic, set.

Remark 2.3. A more typical approach would be to define n-genericity via the meeting or avoiding of all sets that are Σ^0_n *relative to the complexity of the forcing partial order*. (See, e.g., [13], Definition 3.2.7.) For our purposes, n-genericity in this sense corresponds to $(n+2)$-genericity according to Definition 2.2, and as such is a distinction in notation only. We prefer our definition because it will make clearer the connections between Mathias and Cohen genericity that we establish in the sequel, particularly Theorem 5.2.

The following proposition is the analogue of Lemma 2.6 (i) of [7]. The proof is essentially the same, but some small care needs to be taken since the set of conditions here is not computable.

Proposition 2.4. *For each $n \geq 2$, there is a Mathias n-generic G with $G' \leq_T \emptyset^{(n)}$.*

Proof. Let $(D_0, E_0) = (\emptyset, \omega)$, and suppose that we have defined (D_e, E_e) for some $e \geq 0$. First, let $(D^*, E^*) \leq (D_e, E_e)$ be a finite extension that forces the jump, which is straightforward. Then consider $W_e^{\emptyset^{(n-1)}}$. Ask if all pre-conditions (D, E) in this set have E infinite, which is a Π^0_n question since $n \geq 2$. If so, and if there is some pre-condition $(D, E) \leq (D^*, E^*)$ in this set, let (D_{e+1}, E_{e+1}) be some such extension. Otherwise, let $(D_{e+1}, E_{e+1}) = (D^*, E^*)$. In the end, $G = \bigcup_e D_e$ is the desired generic. $\square$

We pass to some other basic properties of generics. We refer to Mathias n-generics below simply as n-generics when no confusion is possible.

Note that the set of all conditions is Π^0_2. Thus, the set of conditions satisfying a given Σ^0_n definition is Σ^0_n if $n \geq 3$, and Σ^0_3 otherwise. For $n < 3$, we may thus wish to consider the following stronger form of genericity, which has no analogue in the case of Cohen forcing (or forcing in general).

Definition 2.5. A set G is *strongly n-generic* if, for every Σ^0_n-definable set of pre-conditions $\mathcal{P}$, either G satisfies some condition in $\mathcal{P}$ or G meets the set of conditions not extended by any condition in $\mathcal{P}$.

Proposition 2.6. *For $n \geq 3$, a set is strongly n-generic if and only if it is n-generic. For $n \leq 2$, a set is strongly n-generic if and only if it is 3-generic.*

Proof. Evidently, every strongly n-generic set is n-generic. Now suppose $\mathcal{P}$ is a Σ^0_n set of pre-conditions, and let $\mathcal{C}$ consist of all the conditions in $\mathcal{P}$. An infinite set meets or avoids $\mathcal{P}$ if and only if it meets or avoids $\mathcal{C}$, so every $\max\{n, 3\}$-generic set meets or avoids $\mathcal{P}$. For $n \geq 3$, this means that every n-generic set is strongly n-generic, and for $n \leq 2$ that every 3-generic set is strongly n-generic.

It remains to show that every strongly 0-generic set is 3-generic. Let $\mathcal{C}$ be a given Σ^0_3 set of conditions, and let R be a computable relation such that (D, E) belongs to $\mathcal{C}$ if and only if $(\exists a)(\forall x)(\exists y)R(D, E, a, x, y)$. Define a strictly increasing computable function g by

$$\Phi_{g(D,E,a)}(x) = \begin{cases} \Phi_E(x) & \text{if } (\exists y)R(D, E, a, x, y) \text{ and } \Phi_E(x)\downarrow, \\ \uparrow & \text{otherwise,} \end{cases}$$

and let $\mathcal{P}$ be the computable set of all pre-conditions of the form $(D, g(D, E, a))$. If $(D, E) \in \mathcal{C}$ then Φ_E is total and so there is an a such that $\Phi_{g(D,E,a)} = \Phi_E$. If, on the other hand, (D, E) is a pre-condition not in $\mathcal{C}$ then for each a there is an x such that $\Phi_{g(D,E,a)}(x)\uparrow$. Thus, the members of $\mathcal{C}$ are precisely the conditions in $\mathcal{P}$, so an infinite set meets or avoids $\mathcal{C}$ if and only if it meets or avoids $\mathcal{P}$. In particular, every strongly 0-generic set meets or avoids $\mathcal{C}$. $\square$

As a consequence, we shall restrict ourselves to 3-genericity or higher from now on, or at most weak 2-genericity. (This is also reasonable from the point of view of Remark 2.3.) Unless otherwise noted, n below will always be a number ≥ 3.

Proposition 2.7. *Every n-generic real is weakly n-generic, and every weakly n-generic real is $(n-1)$-generic.*

Proof. The first implication is clear. For the second, let a Σ^0_{n-1} set $\mathcal{C}$ of conditions be given. Let $\mathcal{D}$ be the set of all conditions that are either in $\mathcal{C}$ or else have no extension in $\mathcal{C}$, which is clearly dense. If $n \geq 4$, then $\mathcal{D}$ is easily seen to be Σ^0_n (actually Π^0_{n-1}) as saying a condition (D, E) has no extension in $\mathcal{C}$ is written

$$\forall (D^*, E^*)[[(D^*, E^*) \text{ is a condition } \wedge (D^*, E^*) \leq (D, E)] \implies (D^*, E^*) \notin \mathcal{C}].$$

If $n = 3$, this makes $\mathcal{D}$ appear to be Σ^0_4 but since $\mathcal{C}$ is a set of conditions only, we can re-write the antecedent of the above implication as

$$D \subseteq D^* \subset D \cup E \wedge (\forall x)[\Phi_{E^*}(x)\downarrow = 1 \wedge \Phi_E(x)\downarrow \implies \Phi_E(x) = 1]$$

to obtain an equivalent Σ^0_3 definition. In either case, then, a weakly n-generic real must meet $\mathcal{D}$, and hence must either meet or avoid $\mathcal{C}$. $\square$

The proof of the following proposition is straightforward. (The first half is proved much like its analogue in the Cohen case. See, e.g., [9], Corollary 2.7.)

Proposition 2.8. *Every weakly n-generic real G is hyperimmune relative to $\emptyset^{(n-1)}$. If G is n-generic, then its degree forms a minimal pair with $\mathbf{0}^{(n-1)}$.*

Corollary 2.9. *Not every n-generic real is weakly $(n+1)$-generic.*

Proof. Take any n-generic $G \leq_T \emptyset^{(n)}$. Then G is not hyperimmune relative to $\emptyset^{(n+1)}$, and so cannot be weakly $(n+1)$-generic. □

We shall separate weakly n-generic reals from n-generic reals in Section 4, thereby obtaining a strictly increasing sequence of genericity notions

$$\text{weakly 3-generic} \leftarrow \text{3-generic} \leftarrow \text{weakly 4-generic} \leftarrow \cdots$$

as in the case of Cohen forcing. In many other respects, however, the two types of genericity are very different. For instance, as noted in [3, Section 4.1], every Mathias generic G is cohesive, i.e., satisfies $G \subseteq^* W$ or $G \subseteq^* \overline{W}$ for every computably enumerable set W. In particular, if we write $G = G_0 \oplus G_1$ then one of G_0 or G_1 is finite. This is false for Cohen generics, which, by an analogue of van Lambalgen's theorem, have relatively n-generic halves (see [16], Proposition 2.2). Thus, no Mathias generic can be even Cohen 1-generic.

Another basic fact is that every Mathias n-generic G is high, i.e., satisfies $G' \geq_T \emptyset''$. (See [1], Corollary 6.7, or [3], Section 5.1 for a proof.) We shall extend this result in Theorem 4.5 below. By contrast, it is a well-known result of Jockusch [7, Lemma 2.6 (ii)] that every Cohen n-generic real G satisfies $G^{(n)} \equiv_T G \oplus \emptyset^{(n)}$. As no high set G can satisfy $G'' \leq_T G \oplus \emptyset''$, it follows that no Mathias generic can have even Cohen 2-generic degree. This does not prevent a Mathias n-generic from having Cohen 1-generic degree, as there are high 1-generic reals. However, we shall show that this does not happen either in Corollary 5.1. However, in Section 5 we shall see that every Mathias n-generic degree bounds a Cohen n-generic degree.

3. Complexity of the forcing relation

Much of the discrepancy between Mathias and Cohen genericity stems from the fact that the complexity of forcing a given arithmetical formula does not agree with the complexity of that formula, as we now show. We work in the usual forcing language, consisting of the language of second-order arithmetic, augmented by a new set constant $\dot{G}$ intended to denote the generic real.

We regard every Σ^0_0 (i.e., bounded quantifier) formula φ with no free number variables as being written in disjunctive normal form according to some fixed effective procedure for doing so. Call a disjunct *valid* if the conjunction of all the equalities and inequalities in it is true, which can be checked computably. For each i (ranging over the number of valid disjuncts), let $P_{\varphi,i}$ be the set of all n such that $n \in X$ is a conjunct of the ith valid disjunct, and $N_{\varphi,i}$ the set of all n such that $n \notin X$ is a conjunct of the ith valid disjunct. Canonical indices for these sets can be determined uniformly effectively from an index for φ.

Definition 3.1. We define the *(strong) forcing relation*, $\Vdash$, for Mathias forcing recursively as follows. Let (D, E) be a condition and let $\varphi(X)$ be a formula with only the set variable X free.

(1) If φ is Σ^0_0, then $(D, E) \Vdash \varphi(\dot{G})$ if for some i, $P_{\varphi,i} \subseteq D$ and $N_{\varphi,i} \subseteq \overline{D \cup E}$.

(2) If $\varphi = \neg\psi$, then $(D, E) \Vdash \varphi(\dot{G})$ if there is no $(D^*, E^*) \leq (D, E)$ such that $(D^*, E^*) \Vdash \psi(\dot{G})$.
(3) If $\varphi = \psi_0 \vee \psi_1$, then $(D, E) \Vdash \varphi(\dot{G})$ if $(D, E) \Vdash \psi_0(\dot{G})$ or $(D, E) \Vdash \psi_1(\dot{G})$.
(4) If $\varphi = (\exists x)\psi(x, X)$ then $(D, E) \Vdash \varphi(\dot{G})$ if $(D, E) \Vdash \psi(n, \dot{G})$ for some $n \in \omega$.

We say (D, E) *forces* $\varphi(\dot{G})$ if $(D, E) \Vdash \varphi(\dot{G})$.

The above definition can also be obtained from a general one for forcing notions in the abstract, by the introduction of a *valuation map*. This is a monotone function V from the forcing partial order into $2^{<\omega}$ such that for each $n \in \omega$, the conditions p with $|V(p)| \geq n$ are dense. (See, e.g., [13, Definition 3.2.3] or [5, Section 3].) For Mathias forcing, the appropriate such V is defined by letting $V((D, E))$ be the string σ of length $\min E$ with $\sigma(x) = 1$ if and only if $x \in D$.

Remark 3.2. If $\varphi(X)$ is Σ^0_0 with only the set variable X free and A is a set then $\varphi(A)$ holds if and only if there is an i such that $P_{\varphi,i} \subseteq A$ and $N_{\varphi,i} \subseteq \overline{A}$. Hence, $(D, E) \Vdash \varphi(\dot{G})$ if and only if $\varphi(D \cup F)$ holds for all finite $F \subset E$.

Lemma 3.3. *Let (D, E) be a condition and let $\varphi(X)$ be a formula in exactly one free set variable.*
(1) *If φ is Σ^0_0 with no free number variables then the relation $(D, E) \Vdash \varphi(\dot{G})$ is computable.*
(2) *If φ is Π^0_1, Σ^0_1, or Σ^0_2, then so is the relation $(D, E) \Vdash \varphi(\dot{G})$.*
(3) *For $n \geq 2$, if φ is Π^0_n then the relation of $(D, E) \Vdash \varphi(\dot{G})$ is Π^0_{n+1}.*
(4) *For $n \geq 3$, if φ is Σ^0_n then the relation $(D, E) \Vdash \varphi(\dot{G})$ is Σ^0_{n+1}.*

Proof. We first prove 1. If φ is as hypothesized and $\varphi(D \cup F)$ does not hold for some finite $F \subset E$, then neither does $\varphi(D \cup (F \cap (\bigcup_i P_{\varphi,i} \cup N_{\varphi,i})))$. So by Remark 3.2, we have that $(D, E) \Vdash \varphi(\dot{G})$ if and only if $\varphi(D \cup F)$ holds for all finite $F \subset E \cap (\bigcup_i P_{\varphi,i} \cup N_{\varphi,i})$, which can be checked computably.

For 2, suppose that $\varphi(X) \equiv (\forall x)\theta(x, X)$, where θ is Σ^0_0. We claim that (D, E) forces $\varphi(\dot{G})$ if and only if $\theta(a, D \cup F)$ holds for all a and all finite $F \subset E$, which makes the forcing relation Π^0_1. The right to left implication is clear. For the other, suppose there is an a and a finite $F \subset E$ such that $\theta(a, D \cup F)$ does not hold. Writing $\theta_a(X)$ for the formula $\theta(a, X)$, let $D^* = D \cup F$ and

$$E^* = \{x \in E : x > \max D \cup F \cup \bigcup_i P_{\theta_a,i} \cup N_{\theta_a,i}\},$$

so that (D^*, E^*) is a condition extending (D, E). Then if (D^{**}, E^{**}) is any extension of (D^*, E^*), we have that

$$D^{**} \cap (\bigcup_i P_{\theta_a,i} \cup N_{\theta_a,i}) = (D \cup F) \cap (\bigcup_i P_{\theta_a,i} \cup N_{\theta_a,i}),$$

and so $\theta(a, D^{**})$ cannot force $\theta(a, \dot{G})$. Thus (D, E) does not force $\varphi(\dot{G})$. The rest of 2 follows immediately, since forcing a formula that is Σ^0_1 over another formula is Σ^0_1 over the complexity of forcing that formula.

We next prove 3 for $n = 2$. Suppose that $\varphi(X) \equiv (\forall x)(\exists y)\theta(x, y, X)$ where θ is Σ^0_0. Our claim is that $(D, E) \Vdash \varphi(\dot{G})$ if and only if, for every a and every condition (D^*, E^*) extending (D, E), there is a finite $F \subset E^*$ and a number $k > \max F$ such that

$$(D^* \cup F, \{x \in E^* : x > k\}) \Vdash (\exists y)\theta(a, y, \dot{G}), \tag{1}$$

which is a Π^0_3 definition. Since the condition on the left side of (1) extends (D^*, E^*), this definition clearly implies forcing. For the opposite direction, suppose $(D, E) \Vdash \varphi(\dot{G})$ and fix any a and $(D^*, E^*) \leq (D, E)$. Then by definition, there is a b and a condition (D^{**}, E^{**}) extending (D^*, E^*) that forces $\theta(a, b, \dot{G})$. Write $\theta_{a,b}(X) = \theta(a, b, X)$, and let $F \subset E^*$ be such that $D^{**} = D^* \cup F$. Since $\theta_{a,b}(D^* \cup F)$ holds, we must have $P_{\theta_{a,b},i} \subseteq D^* \cup F$ and $N_{\theta_{a,b},i} \cap (D^* \cup F) = \emptyset$ for some i. Thus, if we let $k = \max N_{\theta_{a,b},i}$, we obtain (1).

To complete the proof, we prove 3 and 4 for $n \geq 3$ by simultaneous induction on n. Clearly, 3 for $n-1$ implies 4 for n, so we already have 4 for $n = 3$. Now assume 4 for some $n \geq 3$. The definition of forcing a Π^0_{n+1} statement is easily seen to be Π^0_2 over the relation of forcing a Σ^0_n statement, and hence Π^0_{n+2} by hypothesis. Thus, 3 holds for $n+1$. □

We shall see in Corollary 4.2 in the next section that the complexity bounds in parts 3 and 4 of the lemma cannot be lowered to Σ^0_n and Π^0_n, respectively. As a consequence, n-generics only decide all Σ^0_{n-1} formulas, and not necessarily all Σ^0_n formulas.

We conclude this section with a standard result about forcing implying truth. The proof, too, is standard, but relies on the complexity bounds from Lemma 3.3 and some of the particulars of our formalism. Thus, we include the details.

Proposition 3.4. *Let G be n-generic, and for $m \leq n$ let $\varphi(X)$ be a Σ^0_m or Π^0_m formula in exactly one free set variable. If (D, E) is any condition satisfied by G that forces $\varphi(\dot{G})$, then $\varphi(G)$ holds.*

Proof. If $m = 0$, then φ holds of any set satisfying (D, E), whether it is generic or not. If $m > 0$ and the result holds for Π^0_{m-1} formulas, it also clearly holds for Σ^0_m formulas. Thus, we only need to show that if $m > 0$ and the result holds for Σ^0_{m-1} formulas then it also holds for Π^0_m formulas. To this end, suppose $\varphi(X) \equiv (\forall x)\theta(x, X)$, where θ is Σ^0_{m-1}. For each a, let $\mathcal{C}_a$ be the set of all conditions forcing $\theta(a, \dot{G})$, which has complexity at most Σ^0_n by Lemma 3.3. Hence, G meets or avoids each $\mathcal{C}_a$. But if G were to avoid some $\mathcal{C}_a$, say via a condition (D^*, E^*), then (D^*, E^*) would force $\neg\theta(a, \dot{G})$, and then (D, E) and (D^*, E^*) would have a common extension forcing $\theta(a, \dot{G})$ and $\neg\theta(a, \dot{G})$. Thus, G meets every $\mathcal{C}_a$, so $\theta(a, G)$ holds for all a by hypothesis, meaning $\varphi(G)$ holds. □

Remark 3.5. It is not difficult to see that if $\varphi(X)$ is the negation of a Σ^0_m formula then any condition (D, E) forcing $\varphi(\dot{G})$ forces an equivalent Π^0_m formula. Thus, if G is n-generic and satisfies such a condition, then $\varphi(G)$ holds.

4. Jumps of Mathias generic degrees

We begin here with a jump property for Mathias generics similar to the aforementioned one of Jockusch [7, Lemma 2.6 (ii)] for Cohen generics. It follows that the degrees $\mathbf{d}$ satisfying $\mathbf{d}^{(n-1)} = \mathbf{d}' \cup \mathbf{0}^{(n-1)}$ yield a strict hierarchy of subclasses of the high degrees.

Theorem 4.1. *For all $n \geq 2$, if G is n-generic then $G^{(n-1)} \equiv_T G' \oplus \emptyset^{(n)}$.*

Proof. That $G^{(n-1)} \geq_T G' \oplus \emptyset^{(n)}$ follows from the fact that G is high, as discussed above. That $G^{(n-1)} \leq_T G' \oplus \emptyset^{(n)}$ is trivial for $n = 2$. To show it for $n \geq 3$, we wish to decide every $\Sigma^{0,G}_{n-1}$ sentence using $G' \oplus \emptyset^{(n)}$. Let $\varphi_0(X), \varphi_1(X), \ldots,$ be a computable enumeration of all Σ^0_{n-1} sentences in exactly one free set variable, and for each i let $\mathcal{C}_i$ be the set of conditions forcing $\varphi_i(\dot{G})$, and $\mathcal{D}_i$ the set of conditions forcing $\neg\varphi_i(\dot{G})$. Then $\mathcal{D}_i$ is the set of conditions with no extension in $\mathcal{C}_i$, so if G meets $\mathcal{C}_i$ it cannot also meet $\mathcal{D}_i$. On the other hand, if G avoids $\mathcal{C}_i$ then it meets $\mathcal{D}_i$ by definition. Now by Lemma 3.3, each $\mathcal{C}_i$ is Σ^0_n since $n \geq 3$, and so it is met or avoided by G. Thus, for each i, either G meets $\mathcal{C}_i$, in which case $\varphi_i(G)$ holds by Proposition 3.4, or else G meets $\mathcal{D}_i$, in which case $\neg\varphi_i(G)$ holds by Remark 3.5. To conclude the proof, we observe that $G' \oplus \emptyset^{(n)}$ can decide, uniformly in i, whether G meets $\mathcal{C}_i$ or $\mathcal{D}_i$. Indeed, from a given i, indices for $\mathcal{C}_i$ and $\mathcal{D}_i$ (as a Σ^0_n set and a Π^0_n set, respectively) can be found uniformly computably, and then $\emptyset^{(n)}$ has only to produce these sets until a condition in one is found that is satisfied by G, which can in turn be determined by G'. $\square$

Corollary 4.2. *For every $n \geq 2$ there is a Π^0_n formula in exactly one free set variable, the relation of forcing which is not Π^0_n. For every $n \geq 3$ there is a Σ^0_n formula in exactly one free set variable, for which the forcing relation is not Σ^0_n.*

Proof. It suffices to prove the second part, as it implies the first by the proof of Lemma 3.3. For consistency with Theorem 4.1, we fix $n \geq 4$ and prove the result for $n-1$. If forcing every Σ^0_{n-1} formula were Σ^0_{n-1}, then the proof of the theorem could be carried out computably in $G' \oplus \emptyset^{(n-1)}$ instead of $G' \oplus \emptyset^{(n)}$. Hence, we would have $G^{(n-1)} \equiv_T G' \oplus \emptyset^{(n-1)}$, contradicting that G must be high. $\square$

The following result is the analogue of Theorem 2.3 of Kurtz [9] that every $A >_T \emptyset^{(n-1)}$ hyperimmune relative to $\emptyset^{(n-1)}$ is Turing equivalent to the $(n-1)$st jump of a weakly Cohen n-generic real. The proof, although mostly similar, requires a few important modifications. The main problem is in coding A into $G^{(n-2)}$, which, in the case of Cohen forcing, is done by appending long blocks of 1s to the strings under construction. As the infinite part of a Mathias condition can be made very sparse, we cannot use the same idea here. Recall that a set is *co-immune* if its complement has no infinite computable subset. Also, let p_A denote the principal function of the set A, i.e., the function that enumerates the elements of A in increasing order.

Proposition 4.3. *If $A >_T \emptyset^{(n-1)}$ is hyperimmune relative to $\emptyset^{(n-1)}$, then $A \equiv_T G^{(n-2)}$ for some weakly n-generic real G.*

Proof. Computably in A, we build a sequence $(D_0, E_0) \geq (D_1, E_1) \geq \cdots$ of conditions, beginning with $(D_0, E_0) = (\emptyset, \omega)$. Let $\mathcal{C}_0, \mathcal{C}_1, \dots$ be a listing of all Σ^0_n sets of pre-conditions, and fixing a $\emptyset^{(n-1)}$-computable enumeration of each $\mathcal{C}_i$, let $\mathcal{C}_{i,s}$ be the set of all pre-conditions enumerated into $\mathcal{C}_i$ by stage $p_A(s)$. We may assume that $(D, E) \leq s$ for all $(D, E) \in \mathcal{C}_{i,s}$. Let $B_0, B_1, \dots$ be a uniformly $\emptyset^{(n-1)}$-computable sequence of pairwise disjoint co-immune sets. Say $\mathcal{C}_i$ *requires attention* at stage s if there exists $b \leq p_A(s)$ in $B_i \cap E_s$ and a condition (D, E) in $\mathcal{C}_{i,s}$ extending $(D_s \cup \{b\}, \{x \in E_s : x > b\})$.

Construction. At stage s, assume (D_s, E_s) is given. If there is no $i \leq s$ such that $\mathcal{C}_i$ requires attention at stage s, we simply set $(D_{s+1}, E_{s+1}) = (D_s, E_s)$. So suppose otherwise. The construction divides into three steps.

Step 1. Fix the least i such that $\mathcal{C}_i$ requires attention and that we have not yet acted for, and choose the least corresponding b and earliest enumerated extension (D, E) in $\mathcal{C}_{i,s}$. Let $(D^*, E^*) = (D, E)$.

Step 2. Obtain (D^{**}, E^{**}) from (D^*, E^*) by forcing the jump, in the usual manner.

Step 3. Let k be the least bit of A not yet coded. (By construction, this will be the number of stages $t < s$ such that $(D_t, E_t) \neq (D_{t+1}, E_{t+1})$.) Let $(D^{***}, E^{***}) = (D^{**} \cup \{c\}, \{x \in E^{**} : x > c\})$, where c is the least element of $B_{A(k)} \cap E^{**}$, which exists since $B_{A(k)}$ must intersect every computable set infinitely often.

Finally, if $(D^{***}, E^{***}) \leq s+1$ let $(D_{s+1}, E_{s+1}) = (D^{***}, E^{***})$ and say that we have *acted for* the i from Step 1. Otherwise, let $(D_{s+1}, E_{s+1}) = (D_s, E_s)$.

Verification. Clearly, the entire construction is A-computable. To see that $G = \bigcup_s D_s$ is weakly n-generic, fix i and assume that each $\mathcal{C}_j$ with $j < i$ requires attention at most finitely often. Let h be the partial $\emptyset^{(n-1)}$-computable function where $h(s)$ is the least t so that for each (D, E) with $(D, E) \leq s$ there exists $b \leq t$ in $B_i \cap E$ and $(D^*, E^*) \in \mathcal{C}_{i,t}$ extending $(D \cup \{b\}, \{x \in E : x > b\})$. If $\mathcal{C}_i$ is dense then h is total, and so it is infinitely often escaped by p_A. Thus, at some sufficiently large stage, $\mathcal{C}_i$ will require attention and no $\mathcal{C}_j$ with $j < i$ will. We will then act for i under step 1 of some appropriately large subsequent stage, thus ensuring that G meets $\mathcal{C}_i$, and that $\mathcal{C}_i$ never requires attention again.

That $G^{(n-2)} \leq_T A$ follows by Theorem 4.1 from G' being forced at step 2 of the construction, which ensures that $G' \leq_T A$. So it remains only to show that $A \leq_T G^{(n-2)}$. Let $s_0 < s_1 < \cdots$ be all the stages $s > 0$ such that $(D_{s-1}, E_{s-1}) \neq (D_s, E_s)$. (In other words, these are the stages at which code new bits of A.) The sequence $(D_{s_0}, E_{s_0}) > (D_{s_1}, E_{s_1}) \cdots$ can be computed by $G^{(n-2)}$ as follows. Given (D_{s_k}, E_{s_k}), the least $b \in G - D_{s_k}$ must belong to some B_i, and since $G^{(n-2)}$ computes $\emptyset^{(n-1)}$ it can tell which B_i. Then $G^{(n-2)}$ can produce $\mathcal{C}_i$ until it finds the first (D^*, E^*) extending $(D_{s_k} \cup \{b\}, \{x \in E_{s_k} : x > b\})$, and then obtain (D^{**}, E^{**}) from (D^*, E^*) by forcing the jump. By construction, G satisfies (D^{**}, E^{**}) and $(D_{s_{k+1}}, E_{s_{k+1}}) = (D^{**} \cup \{c\}, \{x \in E^{**} : x > c\})$ for the least $c \in G - D_{s_{k+1}}$. And this c is in B_1 or B_0 depending as k is or is not in A. $\square$

Corollary 4.4. *Not every weakly n-generic real is n-generic.*

Proof. By the previous proposition, $\emptyset^{(n)} \equiv_T G^{(n-2)}$ for some weakly n-generic real G. By Theorem 4.1, if G were n-generic we would have $\emptyset^{(n+1)} \equiv_T G^{(n-1)} \equiv_T G' \oplus \emptyset^{(n)} \equiv_T \emptyset^{(n)}$, which cannot be. $\square$

Our final result in this section extends the fact, referenced above, that all Mathias generics are high. Recall that $\mathbf{GH}_1$, the class of *generalized high* degrees, consists of all $\mathbf{d}$ with $\mathbf{d}' = (\mathbf{d} \cup \mathbf{0}')'$. Every degree in this class is obviously high, but the converse in general holds only for degrees $\mathbf{d} \leq \mathbf{0}'$. Note that the usual proof of highness for Mathias genericity proceeds by thinning the infinite parts of conditions so as to eventually dominate all computable functions, and then appealing to Martin's high domination theorem. By contrast, the theorem below is proved by directly using the complexity of the forcing relation.

Theorem 4.5. *If G is n-generic then it has degree in $\mathbf{GH}_1$, i.e., $G' \equiv_T (G \oplus \emptyset')'$.*

Proof. A condition (D, E) forces $i \in (G \oplus \emptyset')'$ if there is a $\sigma \in 2^{<\omega}$ such that that $\Phi_i^\sigma(i)\downarrow$ and for all $x < |\sigma|$,

$$\begin{aligned} \sigma(x) = 1 &\implies (D,E) \Vdash x \in G \oplus \emptyset'; \\ \sigma(x) = 0 &\implies (D,E) \Vdash x \notin G \oplus \emptyset'. \end{aligned}$$

This is thus a Σ_2^0 relation, as forcing $x \in G \oplus \emptyset'$ and $x \notin G \oplus \emptyset'$ are Σ_1^0 and Π_1^0, respectively, by Lemma 3.3. We claim that (D, E) forcing $i \notin (G \oplus \emptyset')'$, i.e., $\neg(i \in (G \oplus \emptyset')')$, is equivalent to (D, E) having no finite extension that forces $i \in (G \oplus \emptyset')'$, and hence is Π_2^0. That forcing implies this fact is clear. In the other direction, suppose (D, E) does not force $i \notin (G \oplus \emptyset')'$, and so has an extension (D^*, E^*) that forces $i \in (G \oplus \emptyset')'$. Let σ witness this fact, as above. Then if P and N consist of the $x < |\sigma|$ such that $\sigma(2x) = 1$ and $\sigma(2x) = 0$, respectively, σ witnesses that $(D \cup P, \{x \in E : x > \max P \cup N\})$ also forces $i \in (G \oplus \emptyset')'$.

We now show that $G' \geq_T (G \oplus \emptyset')'$. Let $\mathcal{C}_i$ be the set of conditions that force $i \in (G \oplus \emptyset')'$, and $\mathcal{D}_i$ the set of conditions that force $i \notin (G \oplus \emptyset')'$. Then $\mathcal{C}_i$ is Σ_3^0 and $\mathcal{D}_i$ is Π_2^0, and indices for them as such can be found uniformly from i. Each $\mathcal{C}_i$ must be either met or avoided by G, and as in the proof of Theorem 4.1, G meets $\mathcal{C}_i$ if and only if it does not meet $\mathcal{D}_i$. Which of the two is the case can be determined by G' since $G' \geq_T \emptyset''$ and $\mathcal{C}_i$ and $\mathcal{D}_i$ are both c.e. in $\emptyset''$. By Proposition 3.4, G' can thus determine whether $i \in (G \oplus \emptyset')'$, as desired. □

5. Relationship with Cohen genericity

We close by directly looking at how Mathias and Cohen generics compare to one another. As remarked at the end of Section 2, no Mathias n-generic set can have Cohen 2-generic degree. Theorem 4.5 above allows us to conclude the same for Cohen 1-genericity. Namely, since no $\mathbf{GH}_1$ degree $\mathbf{d}$ can satisfy $\mathbf{d}' = \mathbf{d} \cup \mathbf{0}'$, it cannot be Cohen 1-generic.

Corollary 5.1. *No Mathias n-generic degree is even Cohen 1-generic.*

Thus, the degree classes of the two types of genericity are disjoint.

In terms of Turing reducibility, rather than equivalence, the situation is more complex. Obviously, for each n there is a Cohen n-generic, namely a $\emptyset^{(n)}$-computable one, that computes no Mathias generic, since the latter generics are all high. In the other direction, recall that $\overline{\mathbf{GL}}_n$ is the class of degrees $\mathbf{d}$ satisfying $\mathbf{d}^{(n)} = (\mathbf{d} \cup \mathbf{0}')^{(n-1)}$. It was shown by Jockusch and Posner [8, Corollary 7] that every $\overline{\mathbf{GL}}_2$ degree bounds a 1-generic degree. Since this class includes every $\mathbf{GH}_1$ degree, it follows by Theorem 4.5 that every Mathias n-generic computes a Cohen 1-generic. For our final result, we show that this can be strengthened to Cohen n-genericity.

Theorem 5.2. *Every Mathias n-generic real computes a Cohen n-generic real.*

Note that we cannot just code each finite string σ by the condition (D, E) with $D = \{x < |\sigma| : \sigma(x) = 1\}$ and $E = \{x : x \geq |\sigma|\}$, and have the generics line up. This would only be the case if the Mathias generic met and avoided all relevant sets by means of conditions of this form, but of course this will not be the case, as conditions of this form are not dense among all the conditions. In fact, we have the following proposition emphasizing this distinction.

Proposition 5.3. *If G is Mathias n-generic and H is Cohen n-generic then H is not many-one reducible to G.*

Proof. Seeking a contradiction, suppose f is a computable function such that $f(H) \subseteq G$ and $f(\overline{H}) \subseteq \overline{G}$. The set of conditions (D, E) with $E \subseteq \operatorname{ran}(f)$ is Σ^0_3-definable, and must be met by G else $G \cap \operatorname{ran}(f)$ would be finite and H would be computable. So fix a condition (D, E) in this set satisfied by G. For all $a > \max D$, we then have that $a \in G$ if and only if $a \in E$ and $f^{-1}(a) \subseteq H$. It follows that $G \leq_T H$, and hence that $G \equiv_T H$, which cannot be. □

Theorem 5.2 is even more surprising because its analog in set theory is known to be false. (In the set-theoretic context, there is no restriction that the infinite part E of a condition (D, E) be computable. Rather, the infinite parts are taken to be elements of a fixed Ramsey ultrafilter.) Indeed, Miller [11, Section 6] showed that adding a Mathias generic real to a transitive model of ZFC does not add a Cohen generic real.

We now proceed to the proof of the theorem.

Proof of Theorem 5.2. We approximate $\emptyset^{(n)}$ by approximating iterations of the jump operator. We fix a uniform way of approximating the jump of a set, and for all $m > 1$, define

$$\emptyset^{(m)}[s_0, \dots, s_{m-1}] = (\emptyset^{(m-1)}[s_0, \dots, s_{m-2}])'[s_{m-1}]$$

by induction. Note that for each x and e, there exist arbitrarily large $s_0, \dots, s_{n-1}$ such that

$$\emptyset^{(n)} \upharpoonright x = \emptyset^{(n)}[s_0, \dots, s_{n-1}] \upharpoonright x$$

and

$$W_e^{\emptyset^{(n)}} \upharpoonright x = W_e^{\emptyset^{(n)}}[s_0, \dots, s_{n-1}] \upharpoonright x = W_e^{\emptyset^{(n)}[s_0,\dots,s_{n-1}]}[s_{n-1}] \upharpoonright x.$$

In fact, we may choose such $s_0, \dots, s_{n-1}$ in any infinite set.

We now wish to define a Turing functional Γ with which to convert Mathias n-generics into Cohen n-generics. For convenience, we regard such a functional as a partial computable map from finite sets under extension to $2^{<\omega}$, with domain closed under initial segment. As is customary, we write Γ^F in place of $\Gamma(F)$, with $\Gamma^F = \tau$ representing that $\Gamma^F(x) \downarrow = \sigma(x)$ for all $x < |\sigma|$, with use bounded by $\max F$.

Construction. Define $\Gamma^{\emptyset} = \emptyset$, and suppose F is a given finite set. Let F_0 be the longest initial segment of F of size a multiple of n, and assume by induction that $\Gamma^{F_0} = \sigma$ has been defined. If the size of F is itself not a multiple of n, set $\Gamma^F = \sigma$. Otherwise, $F = F_0 \cup \{s_0, \dots, s_{n-1}\}$ for some distinct numbers $s_0, \dots, s_{n-1} > \max F_0$. Then let

$$\Gamma^F = \begin{cases} (\mu\tau \succeq \sigma)[\tau \in W_e^{\emptyset^{(n)}}[s_0, \dots, s_{n-1}]] & \text{if } (\exists \tau \succeq \sigma)[\tau \in W_e^{\emptyset^{(n)}}[s_0, \dots, s_{n-1}]], \\ \sigma & \text{otherwise.} \end{cases}$$

Verification. In the construction, we are effectively thinking of each consecutive block of n many elements in F as defining a string. The key observation here is the following. Suppose that $\{s_0, \dots, s_{n-1}\}$ is such a block in F, and that the string σ defined by the previous block has an extension τ in $W_e^{\emptyset^{(n)}}$. Then provided that

$$W_e^{\emptyset^{(n)}} \upharpoonright \tau + 1 = W_e^{\emptyset^{(n)}}[s_0, \dots, s_{n-1}] \upharpoonright \tau + 1, \tag{2}$$

i.e., provided that $s_0, \dots, s_{n-1}$ yields a correct approximation of $W_e^{\emptyset^{(n)}}$ on this τ, then the block $\{s_0, \dots, s_{n-1}\}$ will define an extension of σ in $W_e^{\emptyset^{(n)}}$.

We claim that if G is a Mathias n-generic real then Γ^G is Cohen n-generic. To see this, fix $e \in \omega$, and observe that the set $\mathcal{C}$ of conditions (D, E) such that Γ^D belongs to $W_e^{\emptyset^{(n)}}$ is Σ_n^0-definable. We show that if $\sigma = \Gamma^D$ has an extension in $W_e^{\emptyset^{(n)}}$, then (D, E) has an extension in $\mathcal{C}$. From here the claim follows by genericity, because G meeting or avoiding $\mathcal{C}$ will precisely correspond to Γ^G meeting or avoiding $W_e^{\emptyset^{(n)}}$.

So fix (D, E), and assume $\sigma = \Gamma^D$ has an extension τ in $W_e^{\emptyset^{(n)}}$. By extending (D, E) if necessary, we may assume D has size a multiple of n. Note that this can be done without changing Γ^D. Choose $s_0, \ldots, s_{n-1} \in E$ so that (2) above holds. Then τ belongs to $W_e^{\emptyset^{(n)}}[s_0, \ldots, s_{n-1}]$, so by definition, $\Gamma^{D \cup \{s_0, \ldots, s_{n-1}\}}$ is an extension of σ in $W_e^{\emptyset^{(n)}}$. It follows that

$$(D \cup \{s_0, \ldots, s_{n-1}\}, \{x \in E : x > s_0, \ldots, s_{n-1}\})$$

is an extension of (D, E) in $\mathcal{C}$. This completes the proof. $\square$

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