Reverse Mathematics of Two Theorems of Graph Theory

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2-coloring graphs

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Theorem
Every graph with no cycles of odd length can be 2-colored.
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**Theorem**
Every graph with no cycles of odd length can be 2-colored.

What is the logical strength of this statement?
Reverse Mathematics

**Goal:** Determine exactly which set existence axioms are needed to prove familiar theorems.

**Method:** Prove results of the form

\[ \text{RCA}_0 \vdash \text{AX} \leftrightarrow \text{THM} \]

where:

- **RCA}_0 is a weak axiom system,
- **AX is a set existence axiom selected from a small hierarchy of axioms, and
- **THM is a familiar theorem.
Why bother?

Work in reverse mathematics can:

• precisely categorize the logical strength of theorems.

• differentiate between different proofs of theorems.

• provide insight into the foundations of mathematics.

• utilize and contribute to work in many subdisciplines of mathematical logic – including proof theory, computability theory, models of arithmetic, etc.
RCA₀

Language:
Integer variables: \( x, y, z \) \hspace{1cm} Set variables: \( X, Y, Z \)

Axioms:
- basic arithmetic axioms
  \((0, 1, +, \times, =, \text{and } < \text{ behave as usual.})\)

- Restricted induction
  \((\psi(0) \land \forall n (\psi(n) \rightarrow \psi(n + 1))) \rightarrow \forall n \psi(n)\)
  where \( \psi(n) \) has (at most) one number quantifier.

- Recursive set comprehension
  If \( \theta \in \Sigma^0_1 \) and \( \psi \in \Pi^0_1 \), and \( \forall n (\theta(n) \leftrightarrow \psi(n)) \),
  then there is a set \( X \) such that \( \forall n (n \in X \leftrightarrow \theta(n)) \)
Models and coding

- The smallest \( \omega \)-model of RCA\(_0\) consists of the usual natural numbers and the computable sets of natural numbers. We write \( \mathcal{M} = \langle \omega, \text{REC} \rangle \).
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- RCA$_0$ can prove the arithmetic associated with pairing functions.

- Sets of pairs correspond to functions and/or countable sequences.

- Many mathematical concepts can be encoded in terms of such sequences. Second order arithmetic is remarkably expressive.
Examples

Theorem
(RCA$_0$) Every finite graph with no cycles of odd length can be 2-colored.
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Theorem
\((\text{RCA}_0)\) Every connected graph with no cycles of odd length can be 2-colored.
Weak König’s Lemma

**Statement:** Big very skinny trees are tall.

More formally: If $T$ is an infinite tree in which each node is labeled 0 or 1, then $T$ contains an infinite path.

The subsystem $\text{WKL}_0$ is $\text{RCA}_0$ plus Weak König’s Lemma.

There is an infinite computable $0 - 1$ tree with no infinite computable path, so $\langle \omega, \text{REC} \rangle$ is not a model of $\text{WKL}_0$.

Conclusion: $\text{RCA}_0 \not\vdash \text{WKL}_0$
Finally! Some reverse mathematics!

Theorem

(RCA₀) The following are equivalent:

1. WKL₀.

2. Every graph with no cycles of odd length can be 2-colored.
WKL\(_0\) implies the 2-coloring theorem

Suppose \(G\) is a graph with vertices \(\nu_0, \nu_1, \nu_2, \ldots\) and no odd cycles.
WKL_0 implies the 2-coloring theorem

Suppose $G$ is a graph with vertices $v_0, v_1, v_2, \ldots$ and no odd cycles.

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Let $T$ be the tree consisting of sequences of the form $\langle i_0, i_1, \ldots, i_n \rangle$ where the sequence is a correct 2-coloring of the subgraph of $G$ on the vertices $v_0, v_1, \ldots, v_n$.

Since $G$ has no odd cycles, $\text{RCA}_0$ proves $T$ contains infinitely many nodes.
WKL₀ implies the 2-coloring theorem

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Since $G$ has no odd cycles, RCA₀ proves $T$ contains infinitely many nodes.

Any path through $T$ is the desired 2-coloring.
A tool for reversals

Theorem

(RCA$_0$) *The following are equivalent:*

1. WKL$_0$.

2. *If $f$ and $g$ are injective functions from $\mathbb{N}$ into $\mathbb{N}$ and $\text{Ran}(f) \cap \text{Ran}(g) = \emptyset$, then there is a set $X$ such that $\text{Ran}(f) \subset X$ and $X \cap \text{Ran}(g) = \emptyset$.***

Comment: $X$ in (2) is like a separating set for disjoint computably enumerable sets.
The 2-coloring theorem implies WKL$_0$. A reversal!

Suppose we are given $f$ and $g$ with $\text{Ran}(f) \cap \text{Ran}(g) = \emptyset$.

If, for example, $f(3) = 0$ and $g(2) = 2$, we will construct the graph $G$ as follows:
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Add straight links for $f$ and and shifted links for $g$. 
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Add straight links for \( f \) and and shifted links for \( g \), and 2-color.
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Add straight links for \( f \) and and shifted links for \( g \), and 2-color.
A few other theorems equivalent to WKL$_0$.

Theorem
(RCA$_0$) The following are equivalent:
1. WKL$_0$.
2. Every ctn. function on $[0, 1]$ is bounded. (Simpson)
3. The closed interval $[0, 1]$ is compact. (Friedman)
4. Every closed subset of $\mathbb{Q} \cap [0, 1]$ is compact. (Hirst)
5. Existence theorem for solutions to ODEs. (Simpson)
6. The line graph of a bipartite graph is bipartite. (Hirst)
7. If $\langle x_n \rangle_{n \in \mathbb{N}}$ is a sequence of real numbers then there is a sequence of natural numbers $\langle i_n \rangle_{n \in \mathbb{N}}$ such that for each $j$, $x_{i_j} = \min\{x_n \mid n \leq j\}$. (Hirst)
Arithmetical Comprehension

ACA_0 is RCA_0 plus the following comprehension scheme:

For any formula \( \theta(n) \) with only number quantifiers, the set \( \{ n \in \mathbb{N} \mid \theta(n) \} \) exists.

The minimum \( \omega \) model of ACA_0 contains all the arithmetically definable sets.

Note: WKL_0 \( \not\vdash \) ACA_0, but ACA_0 \( \vdash \) WKL_0.
ACA$_0$ and Graph Theory

Theorem (RCA$_0$) *The following are equivalent:*

1. ACA$_0$
2. *Every graph can be decomposed into its connected components.*
Theorem
(RCA₀) The following are equivalent:

1. ACA₀
2. Every graph can be decomposed into its connected components.

Observation: The proof of “every graph with no odd cycles can be two colored” that starts by decomposing the graph into its connected components makes use of the strong axiom ACA₀. That proof is provably distinct from our proof in WKL₀.
Other theorems equivalent to ACA₀

Theorem
(RCA₀) *The following are equivalent:*

1. ACA₀.
2. *Bolzano-Weierstraß theorem.* (Friedman)
3. *Cauchy sequences converge.* (Simpson)
4. *Ramsey’s theorem for triples.* (Simpson)
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General rule of thumb: ACA₀ suffices for undergraduate math.

RCA₀ proves transfinite induction for arithmetical formulas implies ACA₀. (Hirst)
Other theorems equivalent to $\text{ACA}_0$

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General rule of thumb: $\text{ACA}_0$ suffices for undergraduate math.

RCA$_0$ proves transfinite induction for arithmetical formulas implies $\text{ACA}_0$. (Hirst)

Conclusion: All undergraduate math can be done via transfinite induction arguments.
Ramsey’s theorem on trees

RT$^1$: If $f : \mathbb{N} \rightarrow k$ then there is a $c \leq k$ and an infinite set $H$ such that $\forall n \in H \ f(n) = c$.

TT$^1$: For any finite coloring of $2^{<\mathbb{N}}$, there is a monochromatic subtree order-isomorphic to $2^{<\mathbb{N}}$.

These results extend to colorings of $n$-tuples.
TT$^n_k$ parallels RT$^n_k$

TT$^n_k$: For any $k$ coloring of the $n$-tuples of comparable nodes in $2^{<\mathbb{N}}$, there is a color and a subtree order-isomomorphic to $2^{<\mathbb{N}}$ in which all $n$-tuples of comparable nodes have the specified color.

Note: RT$^n_k$ is an easy consequence of TT$^n_k$

Results in Chubb, Hirst, and McNichol:

- There is a computable coloring with no $\Sigma^0_n$ monochromatic subtree. (Free.)
- Every computable coloring has a $\Pi^0_n$ monochromatic subtree. (Not free.)
- For $n \geq 3$ and $k \geq 2$, RCA$^0_0 \vdash$ TT$^n_k \leftrightarrow$ ACA$^0_0$. 
TT¹ and TT² are problematic

\[ \text{RCA}_0 + \Sigma^0_2 - \text{IND} \text{ can prove } TT^1. \]

\[ \text{RCA}_0 + \text{RT}^1 \text{ does not suffice to prove } TT^1. \]

Corduan, Groszek, and Mileti

Question: Does TT¹ imply \( \Sigma^0_2 - \text{IND} \)?
TT$^1$ and TT$^2$ are problematic

RCA$_0$ + $\Sigma_2^0$ – IND can prove TT$^1$.

RCA$_0$ + RT$^1$ does not suffice to prove TT$^1$.

Corduan, Groszek, and Mileti

Question: Does TT$^1$ imply $\Sigma_2^0$ – IND?

RCA$_0$ + RT$^2$ does not imply ACA$_0$. (Seetapun)

Does RCA$_0$ + TT$^2$ imply ACA$_0$?

Does RCA$_0$ + TT$^2$ imply WKL$_0$?
References


