Research Resources
used in the production of
More reverse mathematics of the Heine-Borel theorem

a joint research project of
Jeff Hirst and Jessica Miller[1]
Department of Mathematical Sciences
Appalachian State University

These slides are available at: www.mathsci.appstate.edu/~jlh

October 21, 2011

[1] Jessica's research was supported by a two semester research assistantship from the **G**raduate **R**esearch **A**ssociate **M**entoring program of the Cratis D. Williams Graduate School.

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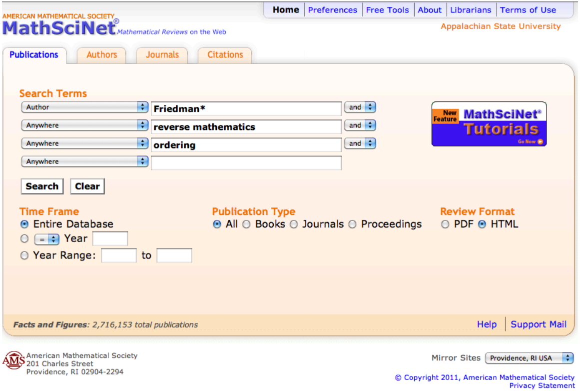

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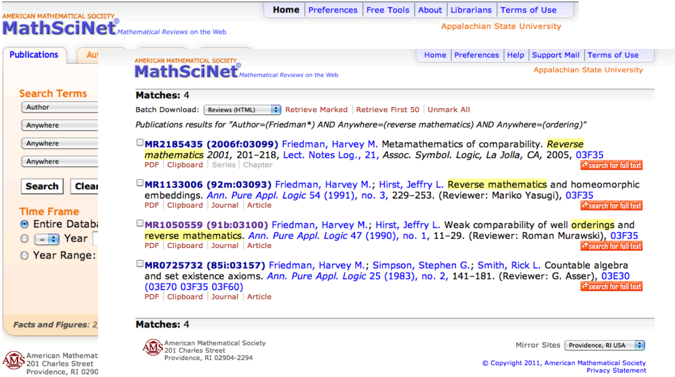

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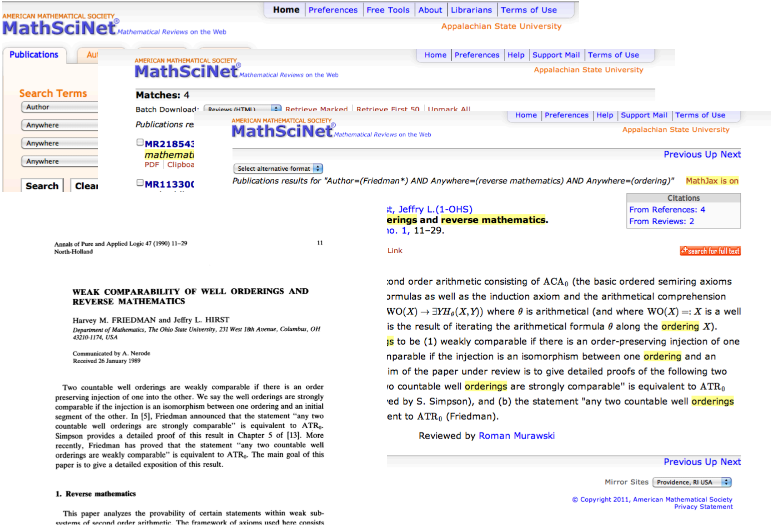

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\noindent
{\it Proof of the claim:}
To prove the reverse implication, suppose that for every $\sigma$, if
$x\in\overline{A}_{\sigma}$ then $x\in\overline{A}_{\sigma\cat 0}$
or $x\in\overline{A}_{\sigma\cat 1}$. Thus we have a sequence of rapidly shrinking neighborhoods
$A_{\sigma_i}$, each of
which contain $x$.  The associated sequence of distinct reals  $\langle x_{\sigma_i} \rangle_{i \in \nat}$
converges to $x$.

To prove the forward implication, we will prove the contrapositive. Suppose
there is a $\sigma$ such that $x\in\overline{A}_{\sigma}$ but $x\notin(\overline{A}_{\sigma\cat 0}\cup\,\overline{A}_{\sigma\cat 1})$.
Either $x=x_{\sigma}$ or $x\neq x_{\sigma}$.  Then either $A_{\sigma}-(\overline{A}_{\sigma\cat 0}\cup\,\overline{A}_{\sigma\cat 1})$
or $P_{\sigma}-(\overline{A}_{\sigma\cat 0}\cup\,\overline{A}_{\sigma\cat 1})$, respectively, is an open set containing $x$ and
no other element of $\{ x_\sigma \mid \sigma \in 2^\nat \}$. So $x$ is not an
accumulation point of $\{ x_\sigma \mid \sigma \in 2^\nat \}$,
completing the proof of the claim.
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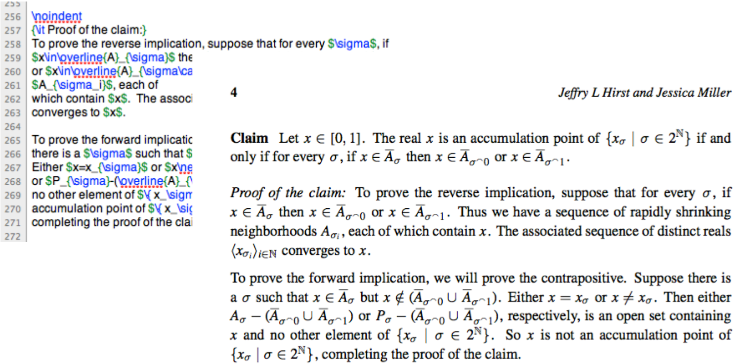

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4 *Jeffry L Hirst and Jessica Miller*

Claim Let $x \in [0,1]$. The real x is an accumulation point of $\{x_\sigma \mid \sigma \in 2^{\mathbb{N}}\}$ if and only if for every σ, if $x \in \overline{A}_\sigma$ then $x \in \overline{A}_{\sigma^\frown 0}$ or $x \in \overline{A}_{\sigma^\frown 1}$.

Proof of the claim: To prove the reverse implication, suppose that for every σ, if $x \in \overline{A}_\sigma$ then $x \in \overline{A}_{\sigma^\frown 0}$ or $x \in \overline{A}_{\sigma^\frown 1}$. Thus we have a sequence of rapidly shrinking neighborhoods A_{σ_i}, each of which contain x. The associated sequence of distinct reals $\langle x_{\sigma_i}\rangle_{i\in\mathbb{N}}$ converges to x.

To prove the forward implication, we will prove the contrapositive. Suppose there is a σ such that $x \in \overline{A}_\sigma$ but $x \notin (\overline{A}_{\sigma^\frown 0} \cup \overline{A}_{\sigma^\frown 1})$. Either $x = x_\sigma$ or $x \neq x_\sigma$. Then either $A_\sigma - (\overline{A}_{\sigma^\frown 0} \cup \overline{A}_{\sigma^\frown 1})$ or $P_\sigma - (\overline{A}_{\sigma^\frown 0} \cup \overline{A}_{\sigma^\frown 1})$, respectively, is an open set containing x and no other element of $\{x_\sigma \mid \sigma \in 2^{\mathbb{N}}\}$. So x is not an accumulation point of $\{x_\sigma \mid \sigma \in 2^{\mathbb{N}}\}$, completing the proof of the claim.

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柳宗元
《漁翁》

漁翁夜傍西巖宿，
曉汲清湘燃楚燭。
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欸乃一聲山水綠。
迴看天際下中流，
巖上無心雲相逐。

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柳宗元
《漁翁》

漁翁夜傍西巖宿，
曉汲清湘燃楚燭。
煙銷日出不見人，
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迴看天際下中流，
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༄༅། །ཡུལ་ཞིག་ན་བྲམ་ཟེ་དབྱུག་པ་ཅན་ཞེས་བྱ་ཞིག་འདུག་སྟེ།
རབ་ཏུ་དབུལ་འཕོངས་པ་བཟའ་བ་དང་། བགོ་མེད་པ་ཞིག་གོ། དེས་
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Andante KV 315

pour flûte et orchestre

W. A. Mozart
transcription pour flûte, hautbois et orgue
D. Taupin

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  title={Notions of closed subsets of a complete separable metric space in
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\bib{frabs}{article}{
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  title={Systems of second order arithmetic with restricted induction, I, II (abstracts)},
  journal={J. Symbolic Logic},
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  }

\bib{fh}{article}{
  author={Friedman, Harvey M.},
  author={Hirst, Jeffry L.},
  title={Reverse mathematics and homeomorphic embeddings},
  journal={Ann. Pure Appl. Logic},
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current formulations seem to have some applicability. For example, Hirst's [4] result showing that the Heine-Borel Theorem for $\mathbb{Q} \cap [0, 1]$ implies WKL_0 is an immediate consequence of Theorem 2. Also, Friedman and Hirst [3] define a canonical embedding of a well ordered set into $[0, 1]$. Working in RCA_0, if we can show that $X \subseteq [0, 1]$ is a countable closed set and is the range of this sort of canonical embedding of a well ordering, then Theorem 1 shows that the Heine-Borel Theorem holds for X.

References

[1] **D K Brown**, *Notions of closed subsets of a complete separable metric space in weak subsystems of second-order arithmetic*, from: "Logic and computation (Pittsburgh, PA, 1987)", Contemp. Math. 106, Amer. Math. Soc., Providence, RI (1990) 39–50

[2] **H M Friedman**, *Systems of second order arithmetic with restricted induction, I, II (abstracts)*, J. Symbolic Logic 41 (1976) 557–559

[3] **H M Friedman**, **J L Hirst**, *Reverse mathematics and homeomorphic embeddings*, Ann. Pure Appl. Logic 54 (1991) 229–253

[4] **J L Hirst**, *A note on compactness of countable sets*, from: "Reverse mathematics

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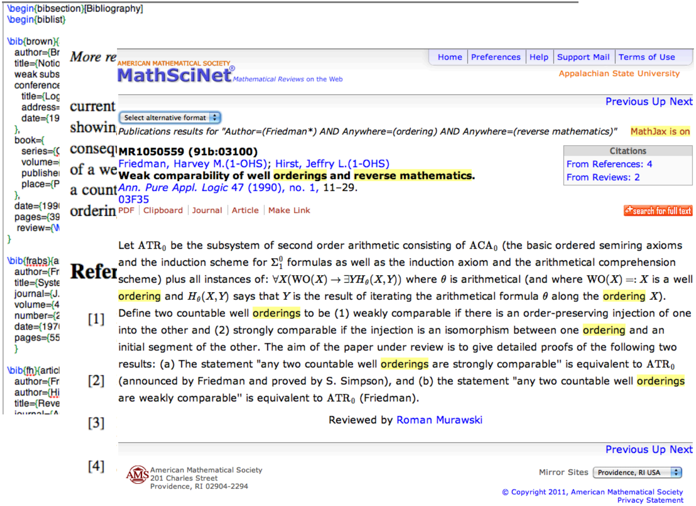

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MR1050559 (91b:03100)
Friedman, Harvey M.(1-OHS); Hirst, Jeffry L.(1-OHS)
Weak comparability of well orderings and reverse mathematics.
Ann. Pure Appl. Logic 47 (1990), no. 1, 11–29.
03F35

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Let ATR_0 be the subsystem of second order arithmetic consisting of ACA_0 (the basic ordered semiring axioms and the induction scheme for Σ^0_1 formulas as well as the induction axiom and the arithmetical comprehension scheme) plus all instances of: $\forall X(\mathrm{WO}(X)\to\exists Y H_\theta(X,Y))$ where θ is arithmetical (and where $\mathrm{WO}(X)=:X$ is a well ordering and $H_\theta(X,Y)$ says that Y is the result of iterating the arithmetical formula θ along the ordering X). Define two countable well orderings to be (1) weakly comparable if there is an order-preserving injection of one into the other and (2) strongly comparable if the injection is an isomorphism between one ordering and an initial segment of the other. The aim of the paper under review is to give detailed proofs of the following two results: (a) The statement "any two countable well orderings are strongly comparable" is equivalent to ATR_0 (announced by Friedman and proved by S. Simpson), and (b) the statement "any two countable well orderings are weakly comparable" is equivalent to ATR_0 (Friedman).

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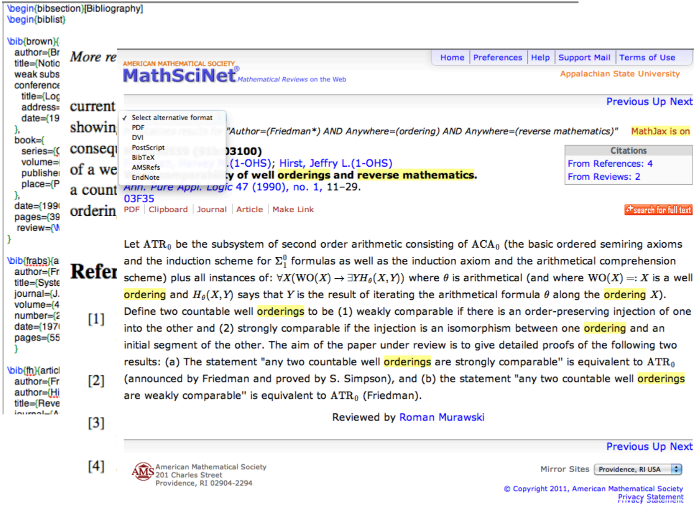

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