Chapter 1

What's So Good About Majority Rule?

Whenever you find yourself on the side of the majority, it is time to reform.
– Mark Twain

Focus Questions

In this chapter, we'll explore the following questions:

- What are some of the systems that can be used to decide the winner of an election with two candidates? What are the strengths and weaknesses of these systems?
- What are some of the criteria that can be used to evaluate voting systems for two-candidate elections?
- What is special or unique about majority rule? Which theorem establishes this uniqueness?
- What is a quota system? How are quota systems related to majority rule and its unique features?

The Mayor of Stickeyville

Warmup 1.1. The time has come for the citizens of Stickeyville to elect a new mayor. Two candidates are running for the office: Mike Dowell and Laura Stutzman. What method should be used to decide the winner?

Did that warmup question seem too easy? Did its answer seem too obvious? If so, perhaps you should consider my proposal for deciding the winner of the election: I have a friend named Stan who lives in Stickeyville. I propose that, to decide the election, the citizens of Stickeyville should all

vote; that's only fair. However, after the voting is done, I think that whoever Stan votes for should be declared the winner, regardless of how any of the other voters vote.

Question 1.2.* Suppose all 101 of the citizens of Stickeyville show up at the polls to vote on election day. If 100 of them vote for Dowell and Stan votes for Stutzman (his girlfriend), who would win the election under the method described in the preceding paragraph?

Your answer to the previous question probably convinced you that the method I proposed for deciding the winner of the Stickeyville mayoral election isn't all that fair. It is, after all, equivalent to a **dictatorship**, which, by definition, is not very democratic. As you might have observed, the main fault of a dictatorship is that it does not treat all of the voters equally. Wouldn't you agree that my method treats Stan (the dictator) in a rather special way?

Allow me to propose another option: Dowell wins, no matter how anyone (including Stan) votes.

Question 1.3.* Does the "Dowell wins" method treat all of the voters equally? Explain.

In spite of your answer to the last question, you probably don't believe that my new proposal is any better than my old one. In fact, the method I suggested—of declaring Dowell the winner no matter how anyone votes—has the decidedly undemocratic name of **imposed rule**. In imposed rule, the outcome is ultimately decided before the election even takes place. Unlike a dictatorship, where at least the dictator's vote matters, with imposed rule nobody's vote matters. Since the winner is decided beforehand, imposed rule suffers from a different problem than a dictatorship: it doesn't treat all of the *candidates* equally. It would be quite an understatement to say that Dowell is favored in the election under imposed rule; in truth, it would be impossible for Dowell to lose, even if everyone voted for Stutzman.

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* Starred questions are intended to help you gauge your understanding of fundamental concepts and definitions before moving on to more difficult material. Partial or complete answers to these questions are provided at the end of each chapter, and we encourage you to use these answers to check your work. However, you should resist the temptation to look ahead to an answer until after you have made a serious attempt to answer the question yourself. You should check the answer only after you are convinced that you have come up with a correct solution on your own. Then if your answer does not agree with the answer at the end of the chapter, you should try to discover what errors you may have made and correct these errors before moving on to subsequent questions. You might want to consider talking to someone else if you need help figuring out how to correct or improve your solutions. It is also important to note that in the answers provided at the end of the chapters, we will often leave out a number of details that you should include when you write your own solutions.

Let me make just one more proposal for deciding the winner of the Stickeyville mayoral election: Each voter should vote for the candidate they want to win the election. The votes for each candidate should then be counted, with the candidate having the *smallest* number of votes declared the winner. Not surprisingly, this method is called **minority rule**.

Question 1.4.* Suppose again that all 101 citizens of Stickeyville show up at the polls, with 100 voting for Dowell and one (Stan) for Stutzman. Who would win the election under minority rule?

Question 1.5. Suppose that Stan convinces 50 of the 100 Dowell voters to change their minds and vote for Stutzman. Who would win the election in this case under minority rule?

Question 1.6. Does minority rule treat all of the voters equally? Does it treat all of the candidates equally? Explain.

Question 1.7. Under minority rule, is it beneficial or detrimental to a candidate to receive additional votes? Explain.

Anonymity, Neutrality, and Monotonicity

So far, we have considered three different methods for deciding the winner of the Stickeyville mayoral election. Such methods are typically called **voting systems**, although it's important to note that this terminology refers not only to the way votes are cast in a particular election, but also to the way in which the winner of the election is determined from the individual votes.

One way to evaluate the fairness of a given voting system is to identify certain desirable properties that we think the system should satisfy, and then see if it actually does satisfy these properties. Recall that we did this for each of the three voting systems we considered in the previous section. In fact, the properties we identified are well known; they have precise names and definitions, which we'll now state.

Definition 1.8.

- A voting system for a two-candidate election is **anonymous** if it treats all of the voters equally, meaning that if any two voters traded votes, the outcome of the election would remain the same.
- A voting system for a two-candidate election is **neutral** if it treats both of the candidates equally, meaning that if *every* voter switched their vote from one candidate to the other, the outcome of the election would change accordingly—the winning candidate would become the losing candidate and the losing candidate would become the winning candidate. (And in the case of a tie, everyone switching their vote would have no effect on the outcome of the election.)

- A voting system for a two-candidate election is **monotone** if it is impossible for a winning candidate to become a losing candidate by gaining votes (and not losing any others) or for a losing candidate to become a winning candidate by losing votes (and not gaining any others).

Question 1.9.* Suppose three children, Zoey, Asher, and Caden, are trying to decide which of their parents, Juan or Melisa, should plan their next family vacation. To make this decision, they decide to hold an election using a voting system invented by their friend Clark (who, incidentally, went to Walt Disney World on his family vacation last year). Three possible combinations of votes by Zoey, Asher, and Caden, and the outcome that Clark's voting system would produce for each combination, are shown in Table 1.1. In the table, J represents a vote for Juan and M a vote for Melisa.

Zoey	Asher	Caden	Winner
J	M	M	J
J	J	M	M
M	M	J	M

TABLE 1.1. Results of Clark's voting system

(a) Which of the three properties described in Definition 1.8 are satisfied by Clark's voting system? Explain.

(b) Is Clark's voting system equivalent to any of the other three voting systems that we've investigated? Why or why not?

Question 1.10.* Suppose you wanted to give an example to convince a friend that a particular voting system is *not* anonymous. According to Definition 1.8, what features would your example need to have?

Question 1.11. Use your answer to Question 1.10 to explain clearly why a dictatorship is not anonymous.

Question 1.12. Explain clearly why a dictatorship is both neutral and monotone.

Question 1.13. Which of the three properties described in Definition 1.8 are satisfied by imposed rule? Which are not satisfied? Give a convincing argument to justify each of your answers.

Question 1.14. Which of the three properties described in Definition 1.8 are satisfied by minority rule? Which are not satisfied? Give a convincing argument to justify each of your answers.

Question 1.15.* Summarize the properties of the three voting systems we've looked at so far by completing the following table.

	Anonymous	Neutral	Monotone
Dictatorship	No	Yes	Yes
Imposed Rule			
Minority Rule			

Majority Rule and May's Theorem

By now, you may feel like we've been dancing around the main point for quite some time. In a sense, you're right. We haven't yet found the perfect voting system for the Stickeyville mayoral election, but we also haven't considered the most obvious choice.

Unless you were trying to be unique, your answer to Warmup 1.1 was probably something like this:

> *Each voter should vote for the candidate they want to win the election. The votes for each candidate should then be counted, and the candidate with the largest number of votes should be declared the winner.*

If you were particularly clever, you might have added something about how ties should be broken. For now, we won't worry about this. We'll just assume that if a tie occurs, it will be broken by some separate procedure.

For two-candidate elections, the voting system described in the preceding paragraph is known as **majority rule**. Majority rule seems on the surface to be much more reasonable than any of the other three systems we've considered. But how does it stand up to the three desirable properties we've discussed—anonymity, neutrality, and monotonicity? As it turns out, majority rule satisfies all three!

Question 1.16. Write a clear and precise explanation of why majority rule is anonymous, neutral, and monotone.

So we've finally found a voting system that behaves nicely, at least according to some reasonably defined standards. The next natural question to ask is this: Are there any other voting systems for two-candidate elections (besides majority rule) that also live up to these standards? The answer to this question comes from mathematician Kenneth May, who proved the following theorem in a paper from 1952 [**35**]:

May's Theorem. *In a two-candidate election with an odd number of voters, majority rule is the only voting system that is anonymous, neutral, and monotone, and avoids the possibility of a tie.*

Question 1.17. In a two-candidate election, why is it especially important for a voting system to avoid the possibility of a tie?

May's Theorem is actually a fairly simple consequence of another theorem regarding *quota systems*, which we'll consider next.

Quota Systems

Definition 1.18. A voting system is called a **quota system** if there is some number q, called the **quota**, such that a candidate will be declared a winner of an election if and only if they receive at least q votes.

A few observations are in order here:

- The phrase "if and only if" has a special meaning in mathematics. In the above definition, it means that
  - if a candidate receives at least q votes, then they will be declared a winner; *and*
  - if a candidate does not receive at least q votes, then they will not be declared a winner.
- In a quota system with two candidates, there could end up being two winners (if both candidates reach the quota) or two losers (if neither candidate reaches the quota). In either case, a separate procedure would have to be used to break the tie.
- In a quota system, the quota can depend on the number of voters casting ballots. For instance, in the state of California, any proposal by a local government to assess a new tax or fee must be approved by two thirds of the electorate. So, if 900 residents voted on such a proposal, then 600 "yes" votes would be required in order for the measure to pass; thus, the quota for the system would be 600. But if 900,000 residents voted, then the quota for the system would be 600,000 instead of 600.

Question 1.19.* Suppose the citizens of Stickeyville decide to use a quota system to elect their new mayor. What would the outcome of the election be in each of the following scenarios?

(a) Quota = 51; Dowell receives 51 votes and Stutzman 50.

(b) Quota = 40; Dowell receives 51 votes and Stutzman 50.

(c) Quota = 60; Dowell receives 51 votes and Stutzman 50.

(d) Quota = 101; Dowell receives 100 votes and Stutzman 1.

(e) Quota = 0; no additional information.

Question 1.20.* Is Clark's voting system from Question 1.9 a quota system? Why or why not?

Question 1.21. Which of the four voting systems that we've studied (dictatorship, imposed rule, minority rule, and majority rule) are quota systems? Give a convincing argument to justify your answer for each system.

We're now ready to investigate the following theorem, of which May's Theorem is a consequence.

Theorem 1.22. *If a voting system for an election with two candidates is anonymous, neutral, and monotone, then it is a quota system.*

Proving a claim like Theorem 1.22 is kind of like solving a mystery. Our suspect, an unknown voting system that we'll call V, has left behind a trail of clues that will eventually allow us to conclude beyond any reasonable doubt that V is actually a quota system. The first three clues are that V is anonymous, neutral, and monotone. We also know that, for any combination of votes in a two-candidate election, V must be able to tell us exactly which candidate (or candidates) it would choose as the winner. (This is, after all, what voting systems do.) Keeping all of this in mind, what we really need to do is ask V the right questions. The answers to these questions will help us identify a value that *could* work as a quota for V. Once we've found this potential quota, say q, our last step will be to argue that V is not just any voting system, but is in fact a quota system with a quota of exactly q.

The next question suggests both the kind of information that we might want to try to extract from V and how we might use this information to determine a potential quota for V.

Question 1.23. Suppose that for an election with two candidates, Jen and Brian, you know the following about V. (Assume that Joel and Grace are just two of many voters in the election.)

- If nobody votes for Jen, then V *will not* choose Jen as a winner.
- If only Joel votes for Jen, then V *will not* choose Jen as a winner.
- If Joel and Grace vote for Jen, then V *will* choose Jen as a winner.

Using only this information and the fact that V is anonymous, neutral, and monotone, could you then say that V is a quota system? If so, what would the quota be? Give a convincing argument to justify your answer, being sure to specify exactly where in your argument you use each of the properties of anonymity, neutrality, and monotonicity. (Hint: You may want to go back and carefully re-read Definition 1.18.)

Question 1.23 demonstrates that once we have extracted the right information, V begins to look a lot like a quota system. Of course, in that question, the information we needed was basically handed to us on a silver platter. We can't expect to always be this lucky, but as we noted earlier, we should be able to find out all the information we need by simply asking V the right questions.

Question 1.24.* Consider an election with two candidates, A and B, and n voters, whom we'll label v_1, v_2, v_3, ..., v_n. (Note that n just represents some arbitrary number of voters.) Suppose that we ask V the following sequence of questions regarding the election:

- If nobody votes for candidate A, would you choose A as a winner?
- If only v_1 votes for candidate A, would you choose A as a winner?
- If v_1 and v_2 vote for candidate A, would you choose A as a winner?
- If v_1, v_2, and v_3 vote for candidate A, would you choose A as a winner?

  ⋮

- If v_1, v_2, v_3, ..., v_{n-1}, and v_n all vote for candidate A, would you choose A as a winner?

Explain how V's answers to these questions could be used to identify a value that might work as a quota for V. Would it ever be possible to identify this potential quota without asking *all* of these questions? Why or why not?

Question 1.25.* Suppose that the method suggested in Question 1.24 was used to identify a potential quota, say q, for V. Clearly explain why each of the following statements would have to be true. Your answers should make use of the fact that V is anonymous, neutral, and monotone.

(a) If exactly q voters (no matter which ones they are) vote for candidate A, then V would choose A as a winner.

(b) If more than q voters (no matter which ones they are) vote for candidate A, then V would choose A as a winner.

(c) If exactly $q - 1$ voters (no matter which ones they are) vote for candidate A, then V would not choose A as a winner.

(d) If fewer than $q - 1$ voters (no matter which ones they are) vote for candidate A, then V would not choose A as a winner.

(e) All of the above conclusions also apply to candidate B.

Question 1.26. Use your answers to Questions 1.24 and 1.25 to clearly explain why Theorem 1.22 is true. That is, explain why in a two-candidate election, every voting system that is anonymous, neutral, and monotone must be a quota system.

Back to May's Theorem

Now that we understand Theorem 1.22 and why it is true, we can finally begin to see why May's Theorem is true. Recall that Theorem 1.22 tells us

that in an election with two candidates, every voting system that is anonymous, neutral, and monotone must be a quota system.[1] May's Theorem tells us that if we assume in addition that the number of voters in the election is odd and that ties are not allowed, then not only must the system be a quota system, but it must be identical to majority rule. Thus, we can prove May's Theorem by arguing that, for a two-candidate election with an odd number of voters, majority rule is the only quota system that avoids ties.

Question 1.27.* Suppose majority rule (a quota system) is used in a two-candidate election with n voters. Describe how to find the quota in this case. (Hint: You know an awful lot about majority rule, so use your intuition, remembering that the quota must be a whole number.)

We'll now argue that, for an odd number of voters, the only quota system that avoids the possibility of ties is the one with the quota that you just found in Question 1.27.

Question 1.28.* Suppose that a two-candidate election is to be decided using a quota system with quota q. Let a and b denote the number of votes received by the two candidates, A and B, respectively.

(a) How must a and b compare to q for the election to result in a tie?

(b) How must a and b compare to q for it to *not* result in a tie?

Question 1.29. Suppose that a two-candidate election with n voters is to be decided using a quota system with quota q.

(a) Suppose q is greater than the quota you found for majority rule in Question 1.27. Give an example to show that, in this case, the outcome of the election could be a tie.

(b) Repeat part (a), but this time assume that q is less than the quota you found for majority rule in Question 1.27.

(c) Suppose n is even and q is exactly equal to the quota you found for majority rule in Question 1.27. Give an example to show that, in this case, the outcome of the election could be a tie.

(d) Suppose n is odd and q is exactly equal to the quota you found for majority rule in Question 1.27. Explain why, in this case, the election *could not* result in a tie.

Question 1.30. Summarize what you learned in this section by writing a clear explanation of how Theorem 1.22 implies May's Theorem. That is, explain why May's Theorem follows from Theorem 1.22.

[1] Incidentally, it's not too hard to see that the backwards version of this statement is also true; that is, every quota system is in fact anonymous, neutral, and monotone.

Question 1.31.

(a) Does there exist a quota system for an election with two candidates that avoids the possibility of ties when there are an even number of voters?

(b) Explain why we need to assume in the statement of May's Theorem that the number of voters is odd.

Questions for Further Study

Question 1.32. In this chapter, we discussed several desirable properties that we might want voting systems to satisfy. Think of another desirable property that we didn't discuss, and explain why you think it would be desirable for voting systems to satisfy this property.

Question 1.33. After delivering a particularly lengthy sermon, the pastor of Long Winds Church is facing a recall vote from his congregation. If two thirds of the voters vote in favor of the recall, then the pastor will become the janitor and the janitor will become the pastor. Otherwise, the pastor will continue to preach and the janitor will continue to mop.

(a) Explain how the procedure for deciding this election could be viewed as a quota system according to the definitions in this chapter.

(b) Suppose that a couple in the church, Greg and Gail, inadvertently cast opposing votes, with Greg voting in favor of the recall and Gail against it. Upon realizing this, Greg remarks to Gail, "Well, I guess our votes cancelled each other out!" Is Greg correct, or would it have been possible for his and Gail's opposing votes to have changed the outcome of the election? Give a convincing argument or example to justify your answer.

Question 1.34. Write a short biography of Kenneth May, including his most important contributions both inside and outside voting theory.

Question 1.35. If there were only two candidates running for president of the United States, would majority rule dictate the winner of the election? Explain.

Question 1.36. If there were only two candidates running for president of the United States, would majority rule dictate the winner of Michigan's electoral votes? What about Nebraska's? Explain.

Question 1.37. Investigate the electoral and popular vote totals received by the candidates in the 1876 U.S. presidential election. Write a summary of your findings, and explain how they relate to our study of majority rule.

Question 1.38.

(a) If the United States Congress voted in an attempt to override a presidential veto, would majority rule dictate whether the override was successful? Explain.

(b) If the United States Congress voted in an attempt to override a presidential veto, would a quota system dictate whether the override was successful? If so, what would the quota for the system be?

Question 1.39. Find a magazine, newspaper, or web site that describes an election with exactly two candidates in which majority rule dictated the winner of the election. Write a detailed summary of your findings.

Question 1.40. Find a magazine, newspaper, or web site that describes an election with exactly two candidates in which majority rule *did not* dictate the winner of the election. Write a detailed summary of your findings.

Question 1.41. Research the pope selection process in the Roman Catholic church. Who are the candidates? Who are the voters? Is the pope selected by majority rule? Write a detailed summary of your findings.

Question 1.42. Consider an election with two candidates, Mya and Luis, and three voters, Ahmed, Beatriz, and Clara. Suppose that if Ahmed and Beatriz vote for Mya, and Clara votes for Luis, then Mya will win. Suppose also that the voting system being used is anonymous, neutral, and monotone. Using only this information, determine what the outcome of the election would be for each of the seven other combinations of votes. Clearly explain your reasoning, including where you used each of the properties of anonymity, neutrality, and monotonicity.

Question 1.43. Consider a voting system for an election with two candidates in which voters from each of two separate parties—say Republican and Democrat—vote separately, and a candidate is declared a winner if and only if they receive more than half of the votes from *each* party. (So, for example, a candidate who received 60% of the Republican votes but only 40% of the Democratic votes would not be declared a winner.)

(a) Is this system anonymous? Give a convincing argument or example to justify your answer.

(b) Is this system neutral? Give a convincing argument or example to justify your answer.

(c) Is this system monotone? Give a convincing argument or example to justify your answer.

Question 1.44. Repeat Question 1.43, but this time assume that a candidate is declared a winner if and only if they receive votes from *more than half* of the Republicans and *less than half* of the Democrats.

Question 1.45. Suppose that in an election with two candidates, a candidate is declared a winner if and only if they receive an even number of votes. Decide whether such a system is anonymous, neutral, and/or monotone. Give a convincing argument or example to justify your answer for each property.

Question 1.46. Research the tie-breaking methods used in various states for general elections. In which state is it possible for the winner to be decided by a game of poker?

Question 1.47. A devious politician has hired you to find or invent a voting system that violates all three of the properties of anonymity, neutrality, and monotonicity. Does such a voting system exist? If so, describe one such system. If not, explain why no such system can exist.

Answers to Starred Questions

1.2. Since Stan's vote is the only one that matters, Stutzman would win.

1.3. The "Dowell wins" method treats all of the voters equally since nobody's vote matters.

1.4. Stutzman would win since she received fewer votes than Dowell.

1.9. (a) None of the three properties are satisfied. The first two rows of Table 1.1 show that Clark's system is not monotone, the first and third rows show that it is not anonymous, and the last two rows show that it is not neutral.

(b) Clark's system is not equivalent to a dictatorship, since neither Zoey, Asher, nor Caden always agree with the winning outcome. It is not equivalent to imposed rule, since the winner is not the same for each combination of votes. And it is not equivalent to minority rule, since in the third row Juan receives fewer votes than Melisa but still loses.

1.10. Your example would need to exhibit two combinations of votes that are the same except for two voters having swapped ballots—and yet the two combinations produce different winners.

1.15. The table can be completed as follows.

	Anonymous	Neutral	Monotone
Dictatorship	No	Yes	Yes
Imposed Rule	Yes	No	Yes
Minority Rule	Yes	Yes	No

1.19. (a) Dowell would win and Stutzman would lose since only Dowell receives at least as many votes as the quota.

(b) Dowell and Stutzman would both win since they both receive at least as many votes as the quota.

(c) Dowell and Stutzman would both lose since they both receive fewer votes than the quota.

(d) Dowell and Stutzman would both lose since they both receive fewer votes than the quota.

(e) Dowell and Stutzman would both win since they would both have to receive at least as many votes as the quota.

1.20. Clark's voting system cannot be a quota system. There are only four possibilities for the quota: 0, 1, 2, and 3. If the quota were 0 or 1, then according to Table 1.1, all three combinations of votes would result in a tie. If the quota were 2, then Melisa would win and Juan would lose in the first row, and Juan would win and Melisa would lose in the second row. And if the quota were 3, then no combination of votes would produce a winner.

1.24. The first question to which V answered yes would indicate the quota for the system. For instance, if V answered yes to the first question, then the quota would be 0, and the system would always result in a tie with both candidates chosen as winners. But if V answered no to the first question but yes to the second, then the quota would be 1. If V answered no to all of the questions, then the quota could be any number greater than n, and the system would always result in a tie with no winner.

1.25. Your answers to parts (a) and (c) should use anonymity, your answers to parts (b) and (d) should use monotonicity, and your answer to part (e) should use neutrality.

1.27. If n is even, then the quota for majority rule is $\frac{n}{2} + 1$. If n is odd, then the quota for majority rule is $\frac{n}{2} + \frac{1}{2}$.

1.28. (a) For the election to result in a tie, a and b would have to both be greater than or equal to q, or a and b would have to both be less than q.

(b) For the election to not result in a tie, exactly one of a and b would have to be greater than or equal to q and the other would have to be less than q.