Chapter 4

Trouble in Democracy

Focus Questions

In this chapter, we'll explore the following questions:

- What is the independence of irrelevant alternatives (IIA) criterion? Which voting systems satisfy this criterion, and which do not?
- What five conditions did Kenneth Arrow think every reasonable voting system should satisfy? What does Arrow's Theorem say about voting systems that satisfy all five of these conditions?
- What are some of the implications of Arrow's Theorem? How is Arrow's Theorem related to the search for a perfect voting system?
- What is Pareto's unanimity condition? How is unanimity related to Arrow's Theorem?

Warmup 4.1. In 1958, Duncan Black, an economist, proposed the following system for deciding the winner of an election with more than two candidates:

- Each voter submits their entire preference order, including all of the candidates in the election.
- If, based on these preference orders, a Condorcet winner exists, then this Condorcet winner is declared the overall winner of the election.
- If no Condorcet winner exists, then the Borda count is used to determine the overall winner of the election.

We have discussed a number of criteria for evaluating voting systems, including anonymity, neutrality, monotonicity, the majority criterion, and the CWC. Which of these criteria does Black's system satisfy, and which does it violate? Explain your answers in detail, and give convincing arguments (including examples where appropriate) to justify your claims.

Independence of Irrelevant Alternatives

So what do you think of Black's system? Before you answer, you might want to consider the following question.

Question 4.2.* Suppose Dale, Paul, and Wayne are the three finalists in the "World's Sexiest Man" contest held aboard the luxury cruise ship, *Floater of the Seas*. Suppose also that the fifteen judges for the contest rank the finalists as shown in the preference schedule in Table 4.1.

	Number of Voters		
Rank	7	6	2
1	P	D	W
2	D	W	P
3	W	P	D

Table 4.1. World's Sexiest Man rankings

(a) Under Black's system, who would win the title of World's Sexiest Man? What ranking (societal preference order) of Dale, Paul, and Wayne would be produced by Black's system?

(b) Suppose that after the votes are cast, but before the winner is announced, Wayne is kicked off the ship for disorderly conduct, thus rendering him ineligible to be a contestant. Given your answer to part (a), should Wayne's exclusion from the contest change its outcome?

(c) Suppose Wayne's name is removed from each of the fifteen ballots shown in Table 4.1, and the remaining contestants are moved up whenever necessary so that each ballot contains only a first- and second-place candidate. What outcome would be produced by Black's system with this new collection of two-candidate ballots?

(d) Does anything about your answer to part (c) strike you as being strange or unusual? Explain.

Question 4.2 shows us that, for all of the strengths of Black's system, it still has one major weakness: the removal of a candidate (Wayne) who stands little or no chance of winning the contest nevertheless has the potential to change its outcome. Because this is true, in this case we would call Wayne a *spoiler candidate*.[1] If we were particularly cynical, we might have even thought that Wayne's very participation in the contest was a sham,

[1] We need only to look back to the 2000 U.S. presidential election (described briefly in Chapter 2) to see this phenomenon in a major political election, with Ralph Nader serving as the potential spoiler candidate. Some have also argued that Gary Johnson and Jill Stein were spoiler candidates in the 2016 U.S. presidential election, and that Hillary Clinton would have won had they not run.

especially if we found out that he was a close friend of Dale's. Of course, it is also possible that Wayne was just overly optimistic about his sexiness. Perhaps he really thought he had a chance of winning the contest and was honestly giving it his best effort. In either case, Wayne's presence or absence in the contest should have been, for all practical purposes, irrelevant to its outcome. And while removing Wayne's name from the ballots didn't change any of the individual judges' orderings of Dale and Paul, it did change the outcome of the contest.

It's also worth pointing out that this sort of behavior can occur and is particularly troubling in situations where a candidate dies before an election, but not in time to be removed from the ballots. Placed in a slightly different context, the example from Question 4.2 shows that a deceased candidate's very presence on the ballots may alter the outcome of an election. And while it is possible for a candidate to be elected post mortem (in which case certain rules would have to be in place to govern the selection of a replacement), what happens far more often is that the candidate becomes a spoiler—not having a serious chance of winning the election, but affecting the outcome nonetheless.

These observations suggest that we might want to add another item to our list of desirable properties that voting systems should satisfy. The property we'll add captures the substance of our discussion above—specifically, that we want voting systems to be unaffected by the presence or absence of irrelevant candidates. One way to formally express this desire is to say that the societal preference between any two candidates should depend only on the voters' preferences *between those two candidates*, and not on the voters' rankings of any of the other candidates. That way, if society prefers candidate A over candidate B, but then candidate C is removed from the election for some reason, society will still prefer A over B. In other words, the societal preference between A and B should not depend at all on where the ineligible choice C might have appeared on each voter's individual preference ballot. That information ought to be irrelevant, just as C is. We formalize this idea in the following definition.

Definition 4.3. If a voting system has the property that the societal preference between any two candidates depends only on the voters' preferences between those two candidates, then the system is said to satisfy the **independence of irrelevant alternatives** criterion (**IIA** for short).

To state Definition 4.3 another way, with a voting system that satisfies IIA, if some or all of the voters in an election change their preference ballots, but no voter changes their preference between two candidates A and B, then the societal preference between A and B must also remain unchanged.

Question 4.4.* Does Black's system satisfy IIA? Why or why not?

Question 4.5. Does the Borda count satisfy IIA? Why or why not?

Question 4.6. Does plurality satisfy IIA? Why or why not?

Question 4.7. Suppose that, in a rematch among the three finalists in the World's Sexiest Man contest, the judges rank Dale, Paul, and Wayne as shown in the preference schedule in Table 4.2.

	Number of Voters		
Rank	5	5	5
1	P	D	W
2	D	W	P
3	W	P	D

TABLE 4.2. World's Sexiest Man rematch

(a) Who would win the contest if the winner was decided by sequential pairwise voting with the agenda D, P, W?

(b) Suppose that, after some private "negotiations," the five judges represented in the rightmost column of Table 4.2 swap the positions of Paul and Dale in their rankings, yielding the new preference schedule shown in Table 4.3. Who would win the contest with this new preference schedule if the winner was again decided by sequential pairwise voting with the agenda D, P, W?

	Number of Voters		
Rank	5	5	5
1	P	D	W
2	D	W	D
3	W	P	P

TABLE 4.3. World's Sexiest Man rematch, revised

(c) What do your answers to parts (a) and (b) allow you to conclude about sequential pairwise voting and IIA? Explain.

Question 4.8. Scoffing at the superficial nature of the World's Sexiest Man contest, the three finalists' wives, Katie, Pam, and Rachel, enter the more civilized "World's Wittiest Woman" contest. Instead of sequential pairwise voting, this contest is to be decided using instant runoff, and the fifteen judges rank Katie, Pam, and Rachel as shown in Table 4.4.

(a) What ranking (societal preference order) of Katie, Pam, and Rachel would be produced from this preference schedule?

(b) In a rematch, the six judges represented in the far left column of Table 4.4 swap the positions of Pam and Rachel in their rankings, yielding the new preference schedule shown in Table 4.5. What ranking would be produced from this new schedule?

	Number of Voters			
Rank	6	3	3	3
1	P	K	R	K
2	R	P	P	R
3	K	R	K	P

TABLE 4.4. World's Wittiest Woman rankings

	Number of Voters			
Rank	6	3	3	3
1	R	K	R	K
2	P	P	P	R
3	K	R	K	P

TABLE 4.5. World's Wittiest Woman rematch

(c) What do your answers to parts (a) and (b) allow you to conclude about instant runoff and IIA? Explain.

Question 4.9.* Of the voting systems we've studied so far—plurality, the Borda count, sequential pairwise voting, instant runoff, and Black's system—which satisfy IIA, and which violate IIA?

Actually, the five voting systems listed in Question 4.9 are not the only ones we have studied. We left out dictatorships, imposed rule, and minority rule, which we discussed back in Chapter 1. These systems each had at least one obvious and serious flaw that caused us to rule them out even for elections with just two candidates. But by now you've surely noticed that we're having a hard time finding a voting system that satisfies the very reasonable list of desirable properties we have constructed. In fact, every voting system we've considered has had at least one notable fault that caused us to keep searching for a better system. You may have felt like throwing in the towel a while back. Or perhaps you're still optimistic that a perfect voting system is out there, and we just need to look a bit harder.

Or perhaps you're a pragmatist. You may reasonably argue that Black's system is the best one we've looked at. It satisfies all of our desirable properties except IIA, and none of the other systems we've looked at satisfy IIA either—or do they? Remember, we haven't yet considered whether dictatorships, imposed rule, and minority rule satisfy IIA.

Question 4.10.*

(a) Do dictatorships satisfy IIA? Why or why not?

(b) Does imposed rule satisfy IIA? Why or why not?

(c) Does minority rule satisfy IIA? Why or why not?

So, as it turns out, there *are* voting systems that satisfy IIA. This shouldn't be too surprising, since IIA does seem to be a pretty reasonable criterion. A far more interesting question is this: Are there voting systems that satisfy IIA *and* some or all of the other desirable properties we've mentioned? For example, is there a voting system that is anonymous, neutral, and monotone, and also satisfies IIA? As we'll see in the next section, the answer to this question is surprising, disturbing, and very significant.

Arrow's Theorem

In 1951, Kenneth Arrow, who was an economist (like Duncan Black), set out on a quest quite similar to the one in which we have been engaged for the last several chapters. Like us, he wanted to find a voting system that was "fair" according to some reasonably defined standards. And like us, he encountered some fairly major roadblocks along the way. Arrow described his experience as follows:

> *I started out with some examples. I had already discovered that these led to some problems. The next thing that was reasonable was to write down a condition that I could outlaw. Then I constructed another example, another method that seemed to meet that problem, and something else didn't seem very right about it. Then I had to postulate that we have some other property. I found I was having difficulty satisfying all of these properties that I thought were desirable... After having formulated three or four conditions of this kind, I kept on experimenting. And lo and behold,* ***no matter what I did, there was nothing that would satisfy these axioms***. [emphasis added]

Does this sound familiar to you? It should! More than half a century later, we've been trying to do exactly the same thing as Arrow, and we've been running into exactly the same problems that he did. So how did Arrow resolve his difficulties? Here's more of what he had to say about his experience:

> *So after a few days of this, I began to get the idea that maybe ... there was no voting method that would satisfy all of the conditions that I regarded as rational and reasonable. It was at this point that I set out to prove it. And it actually turned out to be a matter of only a few days work.*[2]

What Arrow probably didn't realize at the time was that those few days of work would help earn him the 1972 Nobel Prize in economic science, and

[2]Both quotes on this page are from an interview that appears in COMAP [**21**].

that his "impossibility theorem" would come to be regarded as the single most important and well-known result in the history of voting theory. And lest you think that a result of this stature would be comprehensible only to experts, take heart—our investigations throughout the last few chapters have prepared us to understand Arrow's Theorem and even to see why it's true. We'll begin by investigating the definitions and conditions that formed the foundation of Arrow's work.

What is a Voting System?

In Arrow's world, a voting system was a rule that assigned a societal preference order to each possible collection of individual preference orders. To use mathematical language, we could say that a voting system is a *function*; we input into the function the preference orders of all the voters in an election, and the function then spits out an overall ranking of the candidates that in some way represents the will of the electorate.

The fact that, like Arrow, we want voting systems to produce well-defined rankings of the candidates (and not just winners) is very important. For instance, recall from the last chapter that we had problems identifying complete rankings of all the candidates in elections conducted using sequential pairwise voting. The main issue that arose at that time was related to a property that mathematicians call *transitivity*.

In the context of elections, the idea behind transitivity is that if society prefers some candidate A over another candidate B, and also prefers B over a third candidate C, then society ought to prefer A over C. If this is true for any combination of three candidates in an election, then we say that the resulting societal preference order is **transitive**. Recall, however, that the exact opposite happened in Condorcet's paradox (see Question 3.4): society preferred A over B and B over C, but also C over A. In fact, Condorcet's paradox is the classic example of a voting system failing to produce a transitive societal preference order.

Question 4.11.* Suppose X, Y, and Z are the three candidates in an election.

(a) If you know that society prefers X over Z, Z over Y, and X over Y, can you conclude that the resulting societal preference order would be transitive? Explain.

(b) If you know only that society prefers X over Y and Z over X, what would the societal preference between Y and Z have to be in order for the resulting societal preference order to be transitive?

(c) If a fourth candidate entered the election, would your answer to part (a) necessarily be the same? Explain.

Sometimes when societal preference orders fail to satisfy the property of transitivity, we say that the societal preferences represented are *cyclic*. Again, Condorcet's paradox provides a good example of why this wording is appropriate; if we try to combine the results of each of the pairwise comparisons for Question 3.4, the resulting societal preference order will look something like this:

$$A \succ B \succ C \succ A \succ B \succ C \succ A \succ B \succ C \succ A \succ B \succ C \succ \cdots$$

Recall that, in each of the pairwise comparisons that make up this strange societal ranking, the margin of victory was two votes to one. In other words, two thirds of the voters preferred A over B, two thirds preferred B over C, and two thirds preferred C over A.

Question 4.12.* Consider the cyclic societal preferences shown above.

(a) Why might the voters in the election react negatively if A were chosen as the winner of the election? Why might they react negatively if B were chosen as the winner? What about if C were chosen as the winner?

(b) Using your answer to part (a), write a convincing argument for why voting systems that are capable of producing cyclic societal preferences should be avoided.

Given our observations above, from this point forward we will restrict our consideration to voting systems that avoid cyclic societal preferences. That is, we will require the voting systems we consider to produce only transitive societal preference orders. In addition, since we can't expect a voting system to produce something meaningful out of nonsense, we will require the preferences of each of the individual voters to be transitive as well (as has been the case in every example we've considered thus far). With these conditions, we can now formally define a voting system as follows:

Definition 4.13. A **voting system** is a function that receives as input a collection of transitive preference ballots and produces as output a transitive societal preference order.

It's worth noting that nothing in Definition 4.13 rules out ties, either in individual preference ballots or in the societal preference orders produced by a voting system. So, for instance, preference ballots that look like $A \succ B \approx C \succ D$ are perfectly acceptable, as are societal preference orders such as $A \succ B \approx C \approx D$. Thus, with the convention we adopted in Chapter 3, even sequential pairwise voting can be viewed as a voting system in the sense of Definition 4.13.

Arrow's Conditions

Now that we've given a precise definition of what a voting system is, we're ready to move on and state precisely the conditions that Arrow thought every reasonable voting system should satisfy. We'll use the same names that Arrow did in his 1951 book, *Social Choice and Individual Values.*

Condition 1 – Universality. Voting systems should not place any restrictions other than transitivity on how voters can rank the candidates in an election. Specifically, voting systems should not dictate that some preference orders are acceptable while others are not; *every* possible collection of transitive preference ballots must yield a transitive societal preference order.

Condition 2 – Positive Association of Social and Individual Values. Voting systems should be monotone.

Condition 3 – Independence of Irrelevant Alternatives. Voting systems should satisfy IIA.

Condition 4 – Citizen Sovereignty. Voting systems should not be imposed in any way. That is, there should never be a pair of candidates, say A and B, such that A is always preferred over or tied with B in the resulting societal preference order, regardless of how any of the voters vote.

Condition 5 – Nondictatorship. Voting systems should not be dictatorial. That is, there should never be a particular voter—a *dictator*—such that, for any pair of candidates A and B, if the dictator prefers A over B, then society will also prefer A over B.

Question 4.14. Which of Arrow's five conditions is most closely related to the property of anonymity as we defined it in Chapter 2? Which is most closely related to neutrality?

Note that all but one of Arrow's conditions are quite similar to the desirable properties we've studied already. The one that we haven't yet stated explicitly—but have assumed implicitly—is the first; it merely says that a reasonable voting system ought to let voters vote however they want. After all, we might really like majority rule, but majority rule with the added condition that everyone must vote for candidate A is anything but fair.

With that said, it's important to note that a voting system can violate the condition of universality without explicitly placing any restrictions on the ballots voters are allowed to cast. How can this be? Well, recall that in Definition 4.13, we said that a voting system must always output a *transitive* societal preference order. As we saw earlier, however, some potential voting systems, such as the one that produced Condorcet's paradox, naturally yield cyclic societal preferences when faced with certain collections of individual preferences. To deal with such systems, we have two choices: we can either say that the system in question is not really a voting system according to Definition 4.13 (since it is capable of producing cyclic societal preferences), or we can say that it *is* a voting system, but that it can only produce a transitive societal preference order for certain collections of individual preferences. With this latter option, the system would violate universality only because its set of potential inputs would have to be restricted in order to guarantee transitive societal preferences. The next question illustrates how such a restriction could work.

Question 4.15.

(a) Suppose that, in a three-candidate election between candidates A, B, and C, the only individual preference ballots allowed are $A \succ B \succ C$ and $B \succ A \succ C$. Suppose also that the societal preference order for the election is to be formed by simply combining the results of each of the three possible pairwise comparisons (as we did earlier to produce Condorcet's paradox). Explain why, with only these two ballots allowed, it is impossible for the resulting societal preference order to be cyclic.

(b) Suppose that a third ballot, $C \succ B \succ A$, is also allowed. Could the societal preference order be cyclic in this case? Why or why not?

The Punchline

And now, the moment we've been waiting for—a theorem that is both beautiful from a mathematical standpoint and at the same time devastating to our search for the perfect voting system.

Arrow's Theorem. *For an election with more than two candidates, it is impossible for a voting system to satisfy all five of Arrow's conditions.*

The precise wording of Arrow's Theorem is extremely important. The theorem does *not* tell us that mathematicians and social scientists just haven't yet found a voting system that satisfies all five of Arrow's conditions (but perhaps might someday). It also does not tell us that it will be *really difficult* to find such a system. Instead, what Arrow's Theorem says is it is *impossible* for us or anyone else to do so. Try as we might, we will never find a voting system for an election with more than two candidates that satisfies all five of Arrow's very basic and desirable conditions. In other words, every voting system we could ever discover or invent would necessarily have to violate at least one of these five conditions. We can obviously find many systems that satisfy some of Arrow's conditions—and we have done so already—but we will never be able to find a voting system that satisfies *all* of them.

Question 4.16. Which of Arrow's five conditions do you think is the least important for a voting system to satisfy? Give a convincing argument to justify your answer.

Question 4.17.

(a) In your answer to Question 4.6, you probably gave a specific example to show that plurality does not satisfy IIA. Now use Arrow's Theorem (without a specific example) to give another explanation for why this is true.

(b) Could you use Arrow's Theorem, as you did in part (a), to show that instant runoff does not satisfy IIA? Why or why not?

Question 4.18.

(a) Explain how you know that any voting system that is anonymous, neutral, monotone, and satisfies IIA must necessarily satisfy Arrow's conditions 2–5.

(b) Explain why the statement in part (a) implies that any voting system that is anonymous, neutral, monotone, and satisfies IIA must necessarily violate universality.

Arrow himself did not think it was at all unreasonable to require that voting systems satisfy universality, nor could he conceive of a reasonable voting system that would ever violate monotonicity or IIA. Thus, he interpreted his theorem as saying that "the only methods of passing from individual tastes to social preferences which will be satisfactory and which will be defined for a wide range of sets of individual preferences are either imposed or dictatorial."

Others have interpreted Arrow's Theorem differently, and there has actually been a fair amount of debate about what Arrow's Theorem really means and how it should be interpreted in light of the search for a voting system that is truly fair. In the next chapter, we'll consider some of these other interpretations and investigate some potential resolutions to the difficulties revealed by Arrow's work. But before doing so, let's take a quick look at an important and useful variation of Arrow's Theorem.

Pareto's Unanimity Condition

Arrow's Theorem is a surprisingly strong result, but it can actually be made even stronger. Without altering the truth of the theorem, Arrow's conditions 2 and 4 (monotonicity and citizen sovereignty) can be replaced with the following *unanimity* condition, which is sometimes also referred to as the *Pareto condition*, in honor of Vilfredo Pareto (yes, it rhymes!), an Italian economist and political activist who lived in the late 1800s and early 1900s.

The Pareto Condition – Unanimity. If there is a pair of candidates in an election, say A and B, such that *every* voter in the election prefers A over B, then A should be ranked higher than B in the resulting societal preference order.

Question 4.19.* Suppose that candidate A is selected as the winner of an election. For each of the following scenarios, decide, if possible from the information given, whether the voting system used in the election satisfies or violates unanimity. Clearly explain each of your answers.

(a) Candidate A receives no first-place votes; that is, every voter in the election prefers at least one other candidate over A.

(b) There is a candidate in the election, say B, such that every voter in the election prefers B over A.

Question 4.20.*

(a) Does plurality satisfy unanimity? Why or why not?

(b) Does the Borda count satisfy unanimity? Why or why not?

(c) Does instant runoff satisfy unanimity? Why or why not?

Unanimity, like some of the other desirable properties we've considered, seems at first glace to be very reasonable. In fact, unanimity is such a natural and obvious condition that we might expect it to be satisfied by any voting system we could think of. But, as we saw in Question 4.20, this is not the case. In fact, the stronger form of Arrow's Theorem stated below tells us that any voting system that satisfies unanimity will necessarily violate at least one of Arrow's other conditions.

Arrow's Theorem (Strong Form). *For an election with more than two candidates, it is impossible for a voting system to satisfy Arrow's conditions 1, 3, and 5, and unanimity.*

Stated differently, the strong form of Arrow's Theorem says that every voting system that does not dictate the preferences of voters and is not equivalent to a dictatorship must violate either IIA or unanimity (or both). Moreover, some violations of unanimity—like the one in the next question—can be particularly grievous.

Question 4.21. Consider the preference schedule shown in Table 4.6 for an election with four candidates.

(a) Find an agenda for which candidate D would win the election under sequential pairwise voting.

(b) Clearly explain why your answer to part (a) shows that sequential pairwise voting does not satisfy unanimity.

	Number of Voters		
Rank	1	1	1
1	*A*	*B*	*C*
2	*B*	*C*	*D*
3	*C*	*D*	*A*
4	*D*	*A*	*B*

TABLE 4.6. Sequential pairwise voting and unanimity

(c) How is the violation of unanimity that you observed in this question worse in some sense than those you observed in Question 4.20? Explain.

Concluding Remarks

This chapter dealt a seemingly devastating blow to our search for the perfect voting system. It should have also raised a number of questions, such as:

- How can we prove that something is impossible?
- Are Arrow's conditions as reasonable as they seem at first glance?
- If no voting systems are perfect, which ones are best?
- What are some ways of resolving the difficulties illuminated by Arrow's Theorem?

We'll consider these and other questions in the next chapter.

Questions for Further Study

Question 4.22. In this and the previous two chapters, we've considered five voting systems for elections with more than two candidates: plurality, the Borda count, sequential pairwise voting, instant runoff, and Black's system. How do these systems compare to each other when applied to elections with only two candidates? Explain.

Question 4.23. Is Arrow's Theorem true for elections with only two candidates? If so, explain why. Otherwise, give an example of a voting system for an election with two candidates that satisfies all five of Arrow's conditions.

Question 4.24. Of all the desirable properties for voting systems that we've discussed so far, which are satisfied by dictatorships? By imposed rule? By minority rule?

Question 4.25. For each part below, find or invent a voting system for an election with more than two candidates that satisfies each of the three properties listed.

(a) universality, IIA, unanimity

(b) universality, IIA, nondictatorship

(c) universality, unanimity, nondictatorship

(d) IIA, unanimity, nondictatorship

Question 4.26. Find an article about Arrow's Impossibility Theorem in a popular media source. Write a summary and critique of the article based on what you have learned in this chapter.

Question 4.27. Write a short biography of Kenneth Arrow, including his most important contributions both inside and outside of voting theory.

Question 4.28. Write a short biography of Duncan Black, including his most important contributions both inside and outside of voting theory.

Question 4.29. Write a short biography of Vilfredo Pareto, including his most important contributions both inside and outside of voting theory, some information about his political views, and some of the personal problems he faced.

Question 4.30. Black's voting system is an example of a *Condorcet completion system*, meaning that it elects a Condorcet winner if one exists but reverts to some other voting system if one doesn't exist. Research another Condorcet completion system of your own choosing, and write a detailed summary of your findings. Include in your summary a complete evaluation of the system you chose using the criteria we have discussed in this and previous chapters.

Question 4.31. Suppose we redefine a voting system to be a rule that assigns to each possible collection of transitive preference ballots a winning candidate or collection of winning candidates (as opposed to assigning a transitive societal preference order). With this new definition, would Arrow's Theorem still apply? Give a convincing argument to justify your answer. (Hint: You may want to look back at our discussion of sequential pairwise voting and societal preference orders beginning on page 41.)

Question 4.32. Find out how judging is conducted for figure skating competitions at the winter Olympic games, and write a summary of your findings. Include in your summary the actual final ranking and numerical figures from the competition at a recent Olympics.

Question 4.33. Many of the ranking systems used to judge figure skating competitions do not satisfy IIA. Find a magazine, newspaper, or web site that includes an actual example illustrating such a violation. Write a detailed summary of your findings, including the competition in which the incident occurred, the final outcome of the competition, a complete description of the ranking system used by the judges (you may refer to your answer to Question 4.32 if it is the same), and an explanation for how you know that a violation of IIA occurred.

Question 4.34. Consider the following voting system for an election with more than two candidates: Each possible pair of candidates is compared in a head-to-head contest, with one point awarded to the winner, or one-half point to each in the case of a tie. After all the head-to-head comparisons have been completed, the candidate who has been awarded the largest total number of points (or candidates in the case of a tie) is declared the overall winner of the election. This system is often called the *method of pairwise comparisons.*

(a) If the method of pairwise comparisons were used for the PU mathematics chair election from Question 3.17, who would the winner be?

(b) Describe the natural way to construct societal preference orders using the method of pairwise comparisons. Then find the societal preference order produced by the method of pairwise comparisons for the CVAAB presidential election from Question 2.8.

(c) How many head-to-head comparisons would be required for the method of pairwise comparisons in an election with four candidates? What if there were five candidates? What about n candidates (where n represents some arbitrary whole number)?

(d) Describe some of the pros and cons of the method of pairwise comparisons. How does it compare to the other systems we have discussed (plurality, the Borda count, sequential pairwise voting, instant runoff, and Black's system)?

(e) Which of the criteria we have considered (anonymity, neutrality, monotonicity, the majority criterion, the CWC, the CLC, IIA, and unanimity) are satisfied by the method of pairwise comparisons? Which are violated? Explain your answers in detail, and give convincing arguments to justify your claims.

(f) Find a magazine, newspaper, or web site that describes an example where the method of pairwise comparisons was used to arrive at some type of decision or ranking. Write a summary of your findings, including the name of your source and the outcome of the example.

Question 4.35. Research the voting system used by the hit TV show *American Idol*, and write a detailed summary of your findings. Include in your summary a comparison of this voting system to the other systems we have investigated, an analysis of this voting system according to the fairness criteria we have developed, and a discussion of some of the controversy surrounding the voting system.

Question 4.36. Arrow's Theorem was the first of several impossibility theorems to be proved during the second half of the twentieth century. Other well-known results from this time period include Sen's Theorem (also known as Sen's paradox) and the Duggan-Schwartz Theorem. Investigate each of these theorems, and write a detailed summary of your findings. Include in your summary a description and critique of the conditions used in each theorem, how these conditions are related to Arrow's conditions, and a brief biography of the individual(s) for whom each theorem is named.

Question 4.37. In Chapter 3 (Question 3.14), we argued that sequential pairwise voting does not satisfy neutrality. Does sequential pairwise voting satisfy citizen sovereignty? Give a convincing argument or example to justify your answer.

Question 4.38. Find a voting system that satisfies Arrow's nondictatorship condition but is not anonymous, or explain why no such system exists.

Question 4.39. Research each of the following voting systems, and write a detailed summary of your findings. For each system, include in your summary a brief description of the system, an example to illustrate it, and a discussion of which fairness criteria are satisfied by the system.

(a) The Kemeny-Young method

(b) The minimax method (also called Simpson's method)

(c) The Schulze method (also called Schwartz sequential dropping)

(d) The ranked pairs method (also called Tideman's method)

What do all of these systems have in common?

Question 4.40. Research each of the following criteria for evaluating voting systems, and relate them to the other criteria we have discussed so far. Which of these criteria seem most important to you?

(a) The Smith criterion

(b) Reinforcement

(c) Reversal symmetry

(d) Independence of clones

(e) Consistency

(f) Participation

Answers to Starred Questions

4.2. (a) Black's system would revert to the Borda count. Dale would be selected as the winner, and the resulting societal preference order would be $D \succ P \succ W$.

(b) Since Wayne is ranked last in the societal preference order in part (a), his exclusion from the contest shouldn't change the outcome.

(c) With Wayne excluded from the race, Black's system would select Paul as the winner.

(d) It is indeed strange that excluding the last-place candidate from the ranking in part (a) changed the winner of the contest from Dale to Paul.

4.4. Black's system does not satisfy IIA. The example from Question 4.2 shows this.

4.9. None of the systems listed satisfy IIA.

4.10. Both dictatorships and imposed rule satisfy IIA.

4.11. (a) The pairwise preferences given can yield only one possible societal preference order, $X \succ Z \succ Y$, which is transitive.

(b) In order for the resulting societal preference order to be transitive, it would have to be the case that $Z \succ Y$.

(c) With a fourth candidate included in the election, the information given in part (a) would not be sufficient to determine whether the resulting societal preference order would have to be transitive. We would also need to know how society viewed the new candidate in comparison to each of X, Y, and Z.

4.12. (a) If A were chosen as the winner, two thirds of the voters in the election would prefer C. But if B were chosen as the winner, two thirds of the voters would prefer A. And if C were chosen, two thirds of the voters would prefer B.

4.19. (a) Knowing that A wins without any first-place votes does not allow us to conclude that the voting system violates unanimity.

(b) Since every voter prefers B over A, if the system satisfied unanimity, then B would have to be ranked higher than A in the resulting societal preference order. But this is impossible, of course, if A wins the election. Therefore, in this case, the system violates unanimity.

4.20. (a) Plurality *almost* satisfies unanimity, but not quite. Consider an election with three candidates in which every voter has the preference order $A \succ B \succ C$. In this case, every voter would prefer B over C, but B and C would be tied in the resulting societal preference order since neither would receive any first-place votes.

(b) The Borda count satisfies unanimity. (Can you explain why?)

(c) Instant runoff violates unanimity for the same reason that plurality does.