Chapter 7

One Person, One Vote?

Focus Questions

In this chapter, we'll explore the following questions:

- What is a weighted voting system? What are some common examples of weighted systems?
- How are yes/no voting systems similar to the voting systems we investigated in previous chapters? How are they different?
- What does it mean for a voter in a yes/no voting system to be a dictator or to have veto power? Are these properties equivalent?
- How can the properties of swap and trade robustness be used to characterize yes/no voting systems?

Warmup 7.1. After a particularly unsuccessful advertising campaign, the three shareholders of Captain Ahab's Fish & Chips have called an emergency meeting to decide the fate of their vice president of marketing, Deanne Boomhauer. The number of shares of stock held by each shareholder is shown in Table 7.1.

Shareholder	Shares
Doug	101
Nicholas	97
Elisabeth	2

Table 7.1. The shareholders of Captain Ahab's Fish & Chips

At the start of the meeting, Doug makes a proposal to fire Boomhauer and search for a replacement. A vote is called, and each shareholder is to vote either *yes* or *no* on Doug's proposal.

(a) Would majority rule be an appropriate method to use to determine the outcome of the voting? Why or why not?

(b) What method do you think should be used to determine the outcome of the voting? What do you think would happen to Boomhauer if this method were used?

Back in Chapter 1, we identified anonymity (treating all voters equally) as one of the essential properties that voting systems should satisfy. The idea of "one person, one vote" seems to be inextricably linked to our notion of democracy, and rightfully so. But, as we saw in Warmup 7.1, there are situations in which it is not appropriate for the preferences of all of the voters in an election to carry the same weight. In the corporate world, for example, shareholders who own more stock in a company may rightfully deserve a greater say in the company's operating decisions than those who own less stock. In this case, the inequity seems entirely reasonable—after all, it wouldn't make sense for someone who owns only one or two shares of stock to have as much clout as someone who owns half the company.

In answering part (b) of Warmup 7.1, you may have suggested that each shareholder's vote should be weighted in some way that is proportional to the number of shares of stock they own. Voting systems that operate according to this principle are naturally called *weighted* voting systems. They typically arise in situations in which a yes/no or pass/fail decision is required on some motion or proposal. It's important to note the contrast between this type of election and the elections we've considered in previous chapters. There, we were interested in forming some kind of ranking of candidates for an elected office (or at least in choosing a winning candidate). Here, our primary interest will be in deciding the answer to some yes/no question—for example, "Should Boomhauer be fired and a replacement sought?" That's not to say that weighted voting systems don't arise in other contexts. In fact, the system used to elect the president of the United States—the Electoral College—is a type of weighted voting system, which we'll consider in detail in Chapter 9. For now, however, our focus will be on weighted voting systems within the context of yes/no decisions.

Weighted Voting Systems

Definition 7.2. A **yes/no voting system** is a system used to make a decision on a yes/no question, or *motion*. A **weighted voting system** is a yes/no system characterized by the following:

- A collection of voters.
- A collection of **weights**. In particular, we associate with each voter a positive number called the voter's *weight*, which is understood to be the number of votes controlled by the voter.

- A **quota**. The quota is a positive number q such that a motion will *pass* if the sum of the weights of the voters who vote "yes" on the motion equals or exceeds q (and the motion will *fail* otherwise).

There's a lot going on in Definition 7.2, so let's look back at Warmup 7.1 to put it into a natural context.

Question 7.3.* Suppose that a weighted voting system is to be used to decide the outcome of Doug's motion from Warmup 7.1 (to fire Boomhauer and search for a replacement).

(a) Who would the voters in this system be?

(b) What weight should be assigned to each voter?

(c) What would a reasonable choice for the quota be, and why?

Question 7.4.* For parts (a)–(c) below, use the voters and weights that you specified in the first two parts of Question 7.3.

(a) If the quota was 101, which combinations of voters could cause Doug's motion to pass by voting in favor of it? With this quota, what distinguishes Doug from the other voters?

(b) Repeat part (a), but with a quota of 103.

(c) Repeat part (a), but with a quota of 105.

Question 7.4 illustrates several important features of weighted voting systems—and, by extension, yes/no systems in general. First, notice that in a yes/no voting system, whether a motion passes or fails may depend not necessarily on *how many* voters vote in favor of it, but rather on *which* voters vote in favor of it. This fact motivates the following definition.

Definition 7.5. The following definitions apply to yes/no voting systems.

- A **coalition** is a collection of any number of voters, ranging from no voters (the *empty* coalition) to all of the voters in the system.
- A **winning coalition** is a coalition that can force a motion to pass by voting in favor of it. In other words, if every member of a winning coalition votes in favor of a motion, it will pass—even if every voter outside the coalition votes against it.
- A **losing coalition** is a coalition that cannot force a motion to pass by voting in favor of it. In other words, even if every member of a losing coalition were to vote in favor of a motion, the motion could still fail.
- A **minimal winning coalition** is a winning coalition that would become a losing coalition if any individual voter were removed from it.

- In a weighted voting system, the **weight** of a coalition is the sum of the weights of the voters in the coalition.

It's worth noting that if we were only ever going to be interested in weighted voted systems, we could have defined a winning coalition as *a coalition whose weight is greater than or equal to the quota*, and a losing coalition as *a coalition whose weight is less than the quota.* These phrasings are obviously more concise, but also somewhat limiting; our more general definitions will allow us to talk about winning and losing coalitions even when we are considering yes/no voting systems that are not necessarily weighted.

Question 7.6.* For each part of Question 7.4, list all of the winning coalitions, minimal winning coalitions, and losing coalitions.

Question 7.7. Suppose the quota in part (c) of Question 7.4 was 150 instead of 105. Would this change affect any of the winning or minimal winning coalitions for the system? Why or why not?

Question 7.7 shows that, in some cases, two different weighted voting systems can have the exact same winning coalitions. When this occurs, it is natural to say that the two systems are essentially the same, or, to use a favorite word of mathematicians, *isomorphic.*[1]

Definition 7.8. Two weighted voting systems are said to be **isomorphic** if they have the exact same winning coalitions.

In a weighted voting system with n voters, it is a common to use the notation $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$ to represent the voters and their weights, ordered from largest weight to smallest. We can then use the shorthand $[q : w_1, w_2, \ldots, w_n]$ to describe a weighted voting system with weights $w_1, w_2, \ldots, w_n$ and quota q. So, for example, to describe the system from part (b) of Question 7.4, we would write $[103 : 101, 97, 2]$. With this notation in hand, the next question should help you to better understand what it means for two weighted voting systems to be isomorphic.

Question 7.9.* For each of the following weighted voting systems, list all of the winning coalitions. Then decide which of the systems are isomorphic.

(a) $[4 : 2, 2, 1]$

(b) $[4 : 3, 2, 1]$

[1]As a word of caution, as you become more educated in the language of mathematics, you may be tempted to use words such as this in everyday conversation. For example, when discussing your favorite sorority sisters, who happen to be identical twins: "What, you were talking about Julie? I thought you were talking about Michelle! Oh well, they're isomorphic anyways! Ha, ha, ha!" Be forewarned that, while some of your more mathematically-inclined friends may get a good laugh out of this, others may not appreciate your newfound sense of humor.

(c) $[5 : 3, 2, 1]$

(d) $[5 : 3, 2, 2]$

(e) $[5 : 3, 3, 2]$

Dictators, Dummies, and Veto Power

Now that we've defined the various types of coalitions and explored what it means for two weighted voting systems to be isomorphic, let's revisit Question 7.4 one more time. In part (a) of that question, you probably noticed that Doug was a dictator in the sense that the outcome of the voting would always be identical to however he voted. Put another way, Doug was a dictator because he was present in every winning coalition and absent from every losing coalition.

In part (b), however, Doug had slightly less power. While he could still keep a motion from passing by voting against it, he did not have enough weight to singlehandedly force a motion to pass by voting in favor of it. In this situation, we would say that Doug had *veto power*. Put another way, Doug had veto power in part (b) because he was present in every winning coalition. (Notice though that Doug was *not* a dictator in part (b), since he was also present in the losing coalition {Doug}.)

In part (c), Doug's power was even less; he and Nicholas, despite having different weights, were actually equally powerful. Furthermore, poor Elisabeth was completely powerless, unable to affect the outcome in any way, no matter how she voted. In this situation, the somewhat unfortunate convention is to call Elisabeth a *dummy*. Using the language of coalitions, Elisabeth was a dummy in part (c) because she could be removed from any winning coalition to which she belonged, and the coalition would still be winning. Put another way, Elisabeth was a dummy because she was not present in any minimal winning coalitions.

Definition 7.10. In any yes/no voting system:

- A voter who is present in every winning coalition and absent from every losing coalition is called a **dictator**.
- A voter who is present in every winning coalition is said to have **veto power**.
- A voter who is not present in any minimal winning coalitions is called a **dummy**.

Question 7.11.* For each of the weighted voting systems in Question 7.4, use Definition 7.10 to determine which voters are dictators, which are dummies, and which have veto power.

Question 7.12.

(a) In a yes/no voting system, does every dictator have veto power? Is every voter with veto power a dictator? Give a convincing argument or example to justify each of your answers.

(b) Can a yes/no voting system have more than one dictator? Can a yes/no system have more than one voter with veto power? Give a convincing argument or example to justify each of your answers.

Swap Robustness

So far in this chapter, we have been presented with voting systems that we knew were weighted. But what if we were faced with a yes/no voting system in which it was clear that the voters were not all treated equally, but not clear whether we could find weights and a quota to view the system as a weighted system? To consider this scenario, let's look at a couple of examples.

Question 7.13.* The United Nations Security Council consists of fifteen representatives—one from each of fifteen different countries. Five representatives (from China, France, Russia, the United Kingdom, and the United States) are considered permanent members, and the remaining ten representatives change from year to year, with nonpermanent members serving two-year terms. Passage of a motion in the council requires votes in favor from all five permanent members and at least four nonpermanent members.

(a) Can the voting system used to make decisions on motions in the U.N. Security Council be represented as a weighted system? If so, find weights and a quota for the system. Otherwise, explain why it cannot.

(b) In the voting system used to make decisions on motions in the U.N. Security Council, are any of the fifteen representatives dictators or dummies? If so, which ones? Do any of them have veto power? If so, which ones?

Question 7.14. The mystical land of Psykozia is divided into four states: Bliss, Confusion, Disarray, and Ignorance. The federal government of Psykozia is similar to that of the United States. It is comprised of:

- a four-member Senate, with exactly one senator from each state;
- a five-member House of Representatives, with one representative from each state except the highly populated Bliss, which has two representatives; and
- a president and vice president.

As in the United States, passage of a federal bill in Psykozia requires majority support in both the House and Senate, with the vice president casting a tie-breaking vote in the case of a tie in the Senate. The bill is then passed on to the president, who can either sign the bill into law or veto it. A presidential veto can be overridden by a *supermajority* consisting of at least three senators and four representatives.

(a) List all of the different types of winning coalitions that can occur in Psykozia's federal voting system. (Note: We will allow the vice president to be a member of any coalition, even if the makeup of that coalition is such that the vice president's vote is not needed. For instance, we will count the winning coalition containing all eleven members of Psykozia's federal system separately from the winning coalition containing all members except the vice president, even though the vice president's vote is not needed in the first coalition.)

(b) Do you think Psykozia's federal voting system can be represented as a weighted system? If so, try to find weights and a quota for the system. Otherwise, explain why the system cannot be represented as a weighted system.

(c) Are any members of Psykozia's federal voting system dictators or dummies? If so, which ones? Do any of them have veto power? If so, which ones?

Questions 7.13 and 7.14 are similar, but you probably found the latter to be more challenging. That's because Question 7.13 can be answered by simply finding the right weights and quota. It might take some trial and error to do so, but the task is doable. Question 7.14 is another story. It's a pretty safe bet that you weren't able to find weights and a quota that would work to represent Psykozia's federal system as a weighted voting system. That's because Psykozia's federal system *isn't* a weighted voting system.

But how would we prove this? The direct approach seems pretty challenging—after all, we would have to explain why it is impossible to find weights and a quota that are compatible with the description given in Question 7.14. We've seen (for instance, in Chapters 5 and 6) what these types of impossibility arguments can look like and how involved they can be. So perhaps we should consider some other options.

One possibility would be to reason indirectly. For example, our argument could look something like this:

- All mammals are warm-blooded.
- My pet iguana is not warm-blooded.
- Therefore, my pet iguana is not a mammal.

Of course, we'd want to replace the words *mammal*, *iguana*, and *warm-blooded* with other words (like *weighted voting system* and *Psykozia's federal* *voting system*) that would make sense within the context of our problem. But the idea would be the same.

Since we're interested in determining whether Psykozia's federal voting system can be represented as a weighted system, a good first step would be to identify a feature that all weighted systems must possess—like how we identified above that all mammals have the feature of being warm-blooded. We could then try to explain why Psykozia's federal system does not have this feature. If we were able to do so, then by the same type of reasoning that allowed us to conclude that my pet iguana is not a mammal, we'd be able to conclude that Psykozia's federal system cannot be represented as a weighted system.

The feature of weighted voting systems that we'll use involves looking at the outcomes of "swaps" of voters between winning coalitions.

Definition 7.15.

- Let C_1 and C_2 be any two distinct (but possibly overlapping) coalitions for a yes/no voting system. A one-for-one exchange of a voter from C_1 and a voter from C_2 is called a **swap** between C_1 and C_2. To ensure that the same voter will never appear more than once in a coalition, it is required that neither of the swapped voters belong to both coalitions.
- A yes/no voting system is **swap robust** if every possible swap between winning coalitions always leaves at least one of the coalitions still winning.

Question 7.16.* Consider again the three shareholders of Captain Ahab's Fish & Chips from Warmup 7.1.

(a) List all of the possible swaps between the coalitions {Nicholas} and {Doug, Elisabeth}.

(b) List all of the possible swaps between the coalitions {Doug} and {Doug, Nicholas}.

(c) Suppose the shareholders decide to use a bizarre voting system for which the only winning coalitions are {Nicholas} and {Doug, Elisabeth}. Is this system swap robust? Why or why not?

(d) Suppose now that the shareholders decide to use a new voting system for which the only winning coalitions are {Nicholas}, {Doug, Elisabeth}, {Doug}, and {Elisabeth}. Is this system swap robust? Why or why not?

Question 7.17. Can either of the voting systems from parts (c) and (d) of Question 7.16 be represented as weighted systems? Give a convincing argument to justify your answer.

Let's now investigate more fully how the notion of swap robustness is related to whether or not a yes/no voting system can be viewed as weighted. To do so, we'll consider the effect of swaps on the winning coalitions in an arbitrary weighted voting system.

Question 7.18.* Let V be a weighted voting system, and let C_1 and C_2 be any two winning coalitions for V.

(a) How do the weights of C_1 and C_2 relate to the quota for V?

(b) Suppose a swap is made between C_1 and C_2. How does the sum of the weights of C_1 and C_2 after the swap relate to the sum of the weights of C_1 and C_2 before the swap?

(c) Could both C_1 and C_2 be losing coalitions after the swap? Why or why not?

(d) What do your investigations in parts (a)–(c) allow you to conclude about whether V must be swap robust?

If everything went well in Question 7.18, you discovered that swap robustness is a feature that all weighted voting systems must possess. The following theorem summarizes this fact.

Theorem 7.19. *Every weighted voting system must be swap robust.*

Question 7.20.* Suppose for a particular yes/no voting system, you find a swap between two winning coalitions that causes both to become losing coalitions. What can you conclude about the system?

Question 7.21.* Suppose you determine that a particular yes/no voting system is swap robust. Can you conclude that it must be possible to represent the system as a weighted system? Give a convincing argument or example to justify your answer. (Hint: You may find it helpful to look back at some of the previous questions in this section.)

Question 7.22. In light of the last few questions and Theorem 7.19, what can you now conclude about Psykozia's federal voting system? Is it swap robust? Can it be represented as a weighted voting system? Give a convincing argument to justify each of your answers.

Trade Robustness

In the last section, we saw that every weighted voting system must be swap robust. However, we also saw that the property of being swap robust is not completely equivalent to that of being weighted. As we know from Question 7.21, there might (and, in fact, do) exist yes/no voting systems that are swap robust and yet cannot be represented as weighted systems.

At this point, it would seem natural for us to consider whether we can find a property—perhaps related to or similar to swap robustness—such that any yes/no system satisfying that property could in fact be represented as a weighted system. One possible candidate is the following.

Definition 7.23.

- In a yes/no voting system, an arbitrary exchange of voters (not necessarily one-for-one) among at least two coalitions is called a **trade**.
- A yes/no voting system is **trade robust** if every possible trade among winning coalitions always leaves at least one of the coalitions still winning.

Question 7.24.*

(a) Is every swap a trade? Why or why not?

(b) Is every trade a swap? Why or why not?

(c) If a yes/no voting system is swap robust, must it be trade robust? Why or why not?

(d) If a yes/no voting system is trade robust, must it be swap robust? Why or why not?

As we see from Question 7.24, the property of trade robustness is stronger than that of swap robustness. In other words, it is harder for a yes/no voting system to be trade robust than it is for the system to be swap robust. Thus, although we already know that every weighted voting system must be swap robust, we cannot automatically conclude that every weighted system must be trade robust. This does, however, turn out to be true, and we can prove it using an argument similar to that in Question 7.18.

Question 7.25. Let V be a weighted voting system, and let $C_1, C_2, \ldots, C_n$ be any collection of winning coalitions for V.

(a) How do the weights of $C_1, C_2, \ldots, C_n$ relate to the quota for V?

(b) Suppose a trade is made among $C_1, C_2, \ldots, C_n$. How does the sum of the weights of $C_1, C_2, \ldots, C_n$ after the trade relate to the sum of the weights of $C_1, C_2, \ldots, C_n$ before the trade?

(c) Could $C_1, C_2, \ldots, C_n$ all be losing coalitions after the trade? Why or why not?

(d) What do your investigations in parts (a)–(c) allow you to conclude about whether V must be trade robust?

So we now know that every weighted voting system must be trade robust. But is it also true that every yes/no voting system that is trade robust can be represented as a weighted system? As it turns out, the answer to this question is yes. The following theorem summarizes this important result.

Theorem 7.26. *A yes/no voting system can be represented as a weighted system if and only if it is trade robust.*

Theorem 7.26 was proved in 1992 by mathematicians Alan Taylor and William Zwicker, who noted that the proof of a similar result was established in 1960 by computer scientist C. C. Elgot. Although we won't look at either of these proofs (they are quite challenging!), we will conclude this chapter with one very interesting application of Theorem 7.26.

Question 7.27. The voting system used to amend the Constitution of Canada requires that proposed amendments be approved by at least seven of the ten Canadian provinces, and that the approving provinces contain at least half of the total population of Canada. The distribution of the population of Canada (according to the 2016 Canadian census) is shown in Table 7.2.

Province	Percent
Prince Edward Island	0.4%
Newfoundland and Labrador	1.5%
New Brunswick	2.1%
Nova Scotia	2.6%
Saskatchewan	3.1%
Manitoba	3.6%
Alberta	11.6%
British Columbia	13.2%
Quebec	23.2%
Ontario	38.3%

Table 7.2. Population distribution of Canada, 2016

(a) Make a list of all of the minimal winning coalitions in the voting system used to amend the Constitution of Canada. (Assume that the voters in the system are the ten Canadian provinces.)

(b) Is the voting system used to amend the Constitution of Canada swap robust? Give a convincing argument or example to justify your answer.

(c) Is the voting system used to amend the Constitution of Canada trade robust? Give a convincing argument or example to justify your answer.

(d) What do your answers to parts (b) and (c) allow you to conclude about the voting system used to amend the Constitution of Canada? Explain.

Questions for Further Study

Question 7.28. In the weighted voting system $[101 : 101, 97, 2]$ from part (a) of Question 7.4, how would the distribution of power in the system be affected if Doug decided to sell one of his shares of stock to each of Nicholas and Elisabeth?

Question 7.29. Is the property of weighted voting systems being isomorphic a *transitive* property? That is, if a weighted voting system V_1 is isomorphic to another weighted system V_2, and V_2 is isomorphic to a third weighted system V_3, does V_1 have to be isomorphic to V_3? Give a convincing argument to justify your answer.

Question 7.30. In Definition 7.8, we stated that two weighted voting systems are isomorphic if they have the exact same winning coalitions. Find at least three other properties that isomorphic weighted voting systems would necessarily have in common.

Question 7.31. In Definition 7.10, we defined a dictator in a yes/no voting system as a voter who is present in every winning coalition and absent from every losing coalition. Write a different definition of a dictator in a yes/no voting system that involves the idea of minimal winning coalitions.

Question 7.32.

(a) Is it possible for a yes/no voting system to have a dictator but no dummies? Give a convincing argument or example to justify your answer.

(b) Is it possible for a yes/no voting system to have a dummy but no dictator? Give a convincing argument or example to justify your answer.

Question 7.33.

(a) What do you think it would mean for a yes/no voting system to be monotone? Use your understanding of monotonicity from previous chapters to write a precise definition.

(b) According to your definition from part (a), is every weighted voting system monotone? Give a convincing argument or example to justify your answer.

(c) Suppose that in our definition of a weighted voting system, we had allowed weights to be zero or negative. Would this change your answer to part (b)?

(d) Find a yes/no voting system that is monotone and swap robust, but that cannot be represented as a weighted voting system.

Question 7.34. Write short biographies of Alan Taylor and William Zwicker, including their current academic positions and anything you can find about their views on voting theory and social choice.

Question 7.35.

(a) Research the voting system used for enacting federal laws in the United States, including the exact circumstances under which a bill can be passed. Write a detailed summary of your findings, including descriptions of the minimal winning coalitions for the system.

(b) Can the United States' federal voting system be represented as a weighted voting system? Give a convincing argument or example to justify your answer.

Question 7.36. In the United States' federal voting system (like Psykozia's system from Question 7.14), the president can veto a bill, effectively sending it back for further action or death in the Senate and House. Find out what further action must be taken in the Senate and House in order to avoid the death of a vetoed bill. Also, according to the definitions presented in this chapter, does the president in the United States' federal voting system have veto power? Why or why not?

Question 7.37. Without using either of the ideas of swap robustness or trade robustness, write a convincing argument that Psykozia's federal voting system (from Question 7.14) cannot be represented as a weighted system.

Question 7.38. Investigate the voting systems used over the years in the Council of the European Union, and write a detailed summary of your findings.

Question 7.39. Sybil and Joanne are adjusting to life with their newborn son, Curtis. They are quickly finding that leaving the house is not as simple as it used to be. If Sybil and Joanne want to go somewhere without hiring a babysitter, then Curtis must in some sense approve of the trip. For example, if Curtis is sleeping, ill, or otherwise unable to leave the house, then at least one of Sybil or Joanne must stay home with him. Of course, Curtis is also incapable of leaving the house by himself; if he wants to go to his favorite toy store, he must have the approval (or, perhaps more appropriately, cooperation) of at least one of his parents. Consider this family's process of deciding whether or not to go somewhere as a yes/no voting system, with each member of the family being one of the voters.

(a) List all of the winning coalitions for this system.

(b) Are there any dictators, dummies, or voters with veto power in this system?

(c) Is this system weighted? Why or why not?

Question 7.40. Fred, Wilma, Pebbles, and Bam-Bam are trying to decide whether they should stay in and order pizza for dinner or go out to the Dino Rock cafe. To decide this issue, they use a voting system that has the following winning coalitions:

$$\{F, W\}, \{W, P\}, \{P, B\}, \{F, B\}, \{F, W, P\},$$
$$\{F, W, B\}, \{W, P, B\}, \{F, P, B\}, \{F, W, P, B\}$$

(a) Is this voting system swap robust? Why or why not?

(b) What does your answer to part (a) allow you to conclude about whether the system is weighted or not?

(c) Suppose that, in addition to the winning coalitions specified earlier, {W, B} and {F, P} were also winning coalitions. Would the system then be swap robust?

(d) With the addition of these two winning coalitions, would your answer to part (c) allow you to conclude that the system is weighted?

(e) Is the revised system from part (c) weighted? If so, find weights and a quota for it. Otherwise, explain why it cannot be weighted.

Answers to Starred Questions

7.3. In a weighted system for Warmup 7.1, the voters would be Doug, Nicholas, and Elisabeth. The most obvious choices for the weights would be 101 for Doug, 97 for Nicholas, and 2 for Elisabeth. Since the sum of the weights in the system is 200, a reasonable choice for the quota would be 101.

7.4. (a) Any combination of voters that includes Doug could cause the motion to pass by voting in favor of it. With this quota, the outcome of the voting on any motion would always be identical to however Doug voted.

(b) Any combination of voters that includes Doug and at least one other voter could cause the motion to pass by voting in favor of it. With this quota, Doug could not singlehandedly force a motion to pass by voting in favor of it, but he could force a motion to fail by voting against it.

(c) Both Doug and Nicholas would have to vote in favor of a motion in order for it to pass, and it wouldn't matter how Elisabeth voted. With this quota, Doug and Nicholas would be equally powerful, and Elisabeth would be completely powerless.

7.6. (a) The winning coalitions are {Doug}, {Doug, Nicholas}, {Doug, Elisabeth}, and {Doug, Nicholas, Elisabeth}. Only {Doug} is minimal.

(b) The winning coalitions are {Doug, Nicholas}, {Doug, Elisabeth}, and {Doug, Nicholas, Elisabeth}. Both {Doug, Nicholas} and {Doug, Elisabeth} are minimal.

7.9. (a) The winning coalitions are $\{v_1, v_2\}$ and $\{v_1, v_2, v_3\}$.

(b) The winning coalitions are $\{v_1, v_2\}$, $\{v_1, v_3\}$, and $\{v_1, v_2, v_3\}$.

(c) This system is isomorphic to the one in part (a).

7.11. In part (a), Doug is a dictator and has veto power, and Nicholas and Elisabeth are both dummies.

7.13. The system can be represented as a weighted system. You might be able to find weights and a quota by trial and error. Or, you may want to proceed in a more systematic manner; one way to do so is to suppose each nonpermanent member has a single vote, each permanent member has x votes, and the quota is q. Then since votes in favor of a motion from all five permanent members and just four nonpermanent members should cause the motion to pass, $q \leq 5x+4$. And since votes in favor from just four permanent members and all ten nonpermanent members should cause the motion to fail, $4x + 10 < q$. What can you conclude about x by combining these two inequalities? And then what can you conclude about q?

7.16. (a) There are two possible swaps, Nicholas and Doug, and Nicholas and Elisabeth.

(b) There are no possible swaps.

(c) This system is not swap robust. Swapping Nicholas and Doug leaves both coalitions losing.

7.18. (a) The weights of C_1 and C_2 must both be greater than or equal to the quota for V. Note also that the sum of their weights must then be greater than or equal to twice the quota for V.

(b) The sum of the weights of C_1 and C_2 after the swap would be identical to the sum of the weights of C_1 and C_2 before the swap, since the two coalitions together contain the same voters after the swap as before.

(c) If C_1 and C_2 were both losing coalitions after the swap, then the sum of their weights after the swap would have to be less than twice the quota for V. This contradicts the answers to parts (a) and (b).

7.20. You could conclude that the system could not be represented as a weighted system.

7.21. You could not conclude that it must be possible to represent the system as a weighted system. (You could also not conclude that it must be *impossible* to represent the system as a weighted system.)

7.24. (a) A swap is a special kind of trade.

(b) There are lots of trades that are not swaps. (Can you give an example of one?)

(c) A yes/no voting system that is swap robust need not be trade robust.

(d) A yes/no voting system that is trade robust must be swap robust.