Chapter 8

Calculating Corruption

Power tends to corrupt, and absolute power corrupts absolutely.

– Lord Acton, British historian of the late nineteenth and early twentieth centuries.

Power corrupts; absolute power is really neat!

– Donald Regan, White House Chief of Staff, 1985–1987.

Focus Questions

In this chapter, we'll explore the following questions:

- What are some ways of measuring the power held by each of the voters in a yes/no voting system?
- In a yes/no voting system, what is the difference between critical voters and pivotal voters?
- What are the differences between the Banzhaf power index and the Shapley-Shubik power index?
- What is combinatorics? How can the tools of combinatorics be used to calculate power indices?

Warmup 8.1. Consider again the three shareholders of Captain Ahab's Fish & Chips (from Warmup 7.1) and the decision they must make about the fate of their vice president of marketing, Deanne Boomhauer. For reference, the number of shares of stock held by each shareholder is shown again in Table 8.1. Suppose that after much debate, Doug, Nicholas, and Elisabeth agree to adopt a $[103 : 101, 97, 2]$ weighted voting system to make their final decision on Boomhauer.

Shareholder	Shares
Doug	101
Nicholas	97
Elisabeth	2

Table 8.1. The shareholders of Captain Ahab's Fish & Chips

(a) In this weighted voting system, note that Nicholas's weight is more than 48 times Elisabeth's. Does this mean that Nicholas is more than 48 times as powerful as Elisabeth? If not, then exactly how much more powerful is Nicholas than Elisabeth?

(b) In this system, does Doug have more power than Nicholas? If so, how much more?

(c) What percentage of the total power in the system does Doug have? What about Nicholas? Elisabeth?

In Warmup 8.1, you were asked some fairly specific questions about the amount of power held by each of the voters in a weighted voting system. These types of questions arise naturally whenever we are dealing with voting systems in which the voters are not all treated equally.

How you answered Warmup 8.1 probably depends on how you interpreted the word *power*. What does it mean to be "powerful" in the context of a democratic process? And what does it mean to say that one person in such a process has more power than another? Can we quantify this notion of power in order to make meaningful comparisons between the participants in a political system? If so, how?

Our goal in this chapter is to develop some mathematically precise ways of answering questions such as these. To do so, we'll investigate two different methods for measuring the power held by the voters in a yes/no voting system. Each of these methods is called a **power index** because it assigns to each voter some numerical measure of that voter's power. In addition to the power indices themselves, we'll also discover some new mathematical tools that will help us more easily calculate the distribution of power for a number of interesting examples.

The Banzhaf Power Index

The first power index we'll consider was proposed in 1965 by John F. Banzhaf III. Throughout his career as a lawyer and professor of law, Banzhaf specialized in public interest law and public health. He is most well known for his role in a series of lawsuits against the tobacco and fast-food industries, which you may have read about or seen on television.

Banzhaf's views on how power is distributed among voters in yes/no voting systems were based on his belief that a particular voter is more powerful than another if that voter's membership in winning coalitions is more frequently essential, or *critical*, to keeping the coalitions from being losing coalitions. The specifics of the Banzhaf index are given in the next definition, and the questions that follow will help you better understand exactly how the index works.

Definition 8.2.

- A voter in a winning coalition is said to be **critical** if the voter's withdrawal from the coalition would cause it to become a losing coalition.
- The **Banzhaf power** of a voter in a yes/no voting system is the number of winning coalitions in which the voter is critical.
- The **total Banzhaf power** of a yes/no voting system is the sum of the Banzhaf powers of all of the voters in the system.
- The **Banzhaf index** of a voter in a yes/no voting system is the Banzhaf power of the voter divided by the total Banzhaf power of the system.

Question 8.3.* Consider the weighted voting system from Warmup 8.1.

(a) Make a list of all of the winning coalitions for the system.

(b) In how many of the winning coalitions from part (a) is Doug critical? In how many is Nicholas critical? What about Elisabeth?

(c) Using your answers to part (b), determine the Banzhaf power of each of the three voters in the system. Then calculate the total Banzhaf power of the system.

(d) Using your answer to part (c), calculate the Banzhaf index of each of the three voters in the system.

Question 8.4. How do your answers to parts (c) and (d) of Question 8.3 compare to your answer to Warmup 8.1?

Question 8.5. Which do you think would be more useful to know: the Banzhaf power of a voter or the Banzhaf index of a voter? Explain.

Question 8.6.

(a) What would the Banzhaf index of a dictator be? What about a dummy? Clearly explain your answers using the terms given in Definition 8.2.

(b) What, if anything, can you say about the Banzhaf index of a voter who has veto power? Give a convincing argument to justify your answer.

Now that we understand the basic idea behind the Banzhaf index, we should be ready to look at a slightly more complicated example. In the next question, we'll investigate the actual situation that prompted Banzhaf to develop his index in the first place.

Question 8.7.* The Board of Supervisors in Nassau County, New York used a weighted voting system that gave representation to each of six districts in the county according to their relative populations. In 1965, a total of 115 votes were allocated to the districts, as shown in Table 8.2. For a motion to be passed, a simple majority of the total number of votes was required, and thus the quota for the system was 58.

District	Votes
Hempstead 1	31
Hempstead 2	31
Oyster Bay	28
North Hempstead	21
Long Beach	2
Glen Cove	2

TABLE 8.2. Nassau County Board of Supervisors, 1965

(a) In a series of lawsuits, Banzhaf successfully argued that all of the power in the board was equally distributed among the three largest districts. Without actually calculating the Banzhaf power or index of any of the districts, explain why this was in fact true.

(b) Make a list of all of the winning coalitions for the system.

(c) For each of the winning coalitions from part (b), identify all of the voters that are critical to the coalition.

(d) Using your answer to part (c), determine the Banzhaf index of each of the six districts in the system.

(e) Does your answer to part (d) support Banzhaf's claim that all of the power in the board was equally distributed among the three largest districts? Explain.

Question 8.8.* As a result of Banzhaf's lawsuits, the allocation of votes in the Nassau County Board of Supervisors was changed, and actually changed several times before a federal judge declared the board unconstitutional in 1994. The final allocation of votes in the board in 1994 yielded the weighted voting system $[65 : 30, 28, 22, 15, 7, 6]$, with the districts listed in the same order as how they are listed in Table 8.2.

(a) Following the same steps as in parts (b)–(d) of Question 8.7, determine the Banzhaf index of each of the six districts in the 1994 system.

(b) In the 1994 system, what percentage of the total power did each of the six districts have? What percentage of the total power did each of the six districts have in the 1965 system? Does the 1994 system seem more reasonable than the 1965 system?

(c) Suppose that in 1994, a little less than 2% of the population of Nassau County lived in the Glen Cove district. If this were true, then do you think it was acceptable that Glen Cove was given $\frac{6}{108} = 5.56\%$ of the votes in the Nassau County Board of Supervisors in 1994? Why or why not?

The Shapley-Shubik Power Index

The second power index we'll consider is named for economists Lloyd Shapley and Martin Shubik. Shapley and Shubik first proposed their index in 1954, eleven years before Banzhaf proposed his. We looked at the Banzhaf index first though because the Shapley-Shubik index is somewhat harder to calculate, and the mathematics behind it is slightly more sophisticated.

Shapley and Shubik's views on how power is distributed among voters in yes/no voting systems were based on the idea of *pivotal* voters instead of *critical* voters. Shapley and Shubik believed that coalitions in voting systems were formed sequentially, with some voter joining first, a different voter joining second, another third, and so on. And when it is assumed that members join a winning coalition in some order, it makes sense to talk about which voter first gives the coalition enough total weight to make it a winning coalition. It is this unique voter in an ordered coalition that we will call the *pivotal voter* for the coalition.[1] This terminology is made more precise in the following definition.

Definition 8.9.

- For some arrangement (ordering) of all of the voters in a yes/no voting system, we say that a voter v is **pivotal** if both of the following conditions hold:
  - If each of the voters before v in the arrangement votes for a motion to pass, and v and each of the voters after v votes for the motion to fail, then the motion will fail.
  - If v and each of the voters before v in the arrangement votes for a motion to pass, and each of the voters after v votes for the motion to fail, then the motion will pass.

[1] Note that this definition is consistent with how we used the term *pivotal voter* in our proof of Arrow's Theorem in Chapter 5.

- The **Shapley-Shubik power** of a voter in a yes/no voting system is the number of arrangements of all of the voters in the system in which the voter is pivotal.
- The **total Shapley-Shubik power** of a yes/no voting system is the total number of arrangements of all of the voters in the system.
- The **Shapley-Shubik index** of a voter in a yes/no voting system is the Shapley-Shubik power of the voter divided by the total Shapley-Shubik power of the system.

You might find it interesting to note that the definition of the Shapley-Shubik index never actually mentions winning coalitions. This is because when we want to calculate a Shapley-Shubik index, we typically just consider every possible arrangement of all of the voters in the system, and then identify the unique pivotal voter in each arrangement. We can do this because every sequentially formed coalition will eventually go from losing to winning as voters are added. Thus, by considering all possible arrangements of all of the voters, we will eventually discover all of the (ordered) winning coalitions. Proceeding in this manner is helpful for a number of reasons, one of which is the fact that we can very easily determine the total number of arrangements of all of the voters in a system by using a simple formula that we will discover in a moment.

But first, an example: Let's find the Shapley-Shubik indices of the three shareholders of Captain Ahab's Fish & Chips.

Question 8.10.*

(a) Make a list of every possible arrangement of all of the voters in the weighted voting system from Warmup 8.1.

(b) Identify the pivotal voter in each of the arrangements from part (a).

(c) Using your answer to part (b), determine the Shapley-Shubik index of each of the voters in the weighted voting system from Warmup 8.1.

Question 8.11. Compare your answer to part (c) of Question 8.10 with your answer to part (d) of Question 8.3.

(a) Should these two answers be the same? Are they the same? Explain any discrepancies between the two, and discuss whether you think it's reasonable for these discrepancies to exist.

(b) Which index do you think better represents the actual distribution of power in the weighted voting system from Warmup 8.1? Clearly explain your answer.

Question 8.12.

(a) What would the Shapley-Shubik index of a dictator be? What about a dummy? Clearly explain your answers using the terms given in Definition 8.9.

(b) What, if anything, can you say about the Shapley-Shubik index of a voter who has veto power? Give a convincing argument to justify your answer.

As we saw in Question 8.10, not only are the calculations involved in finding Shapley-Shubik indices different from those involved in finding Banzhaf indices, but the results might be different too. This is analogous to what we saw in previous chapters when we looked at the outcomes produced by different voting systems for elections with more than two candidates.

There are some other significant differences between the two indices as well. For one thing, the Banzhaf power of an individual voter is generally easier to calculate than the Shapley-Shubik power. However, the total Banzhaf power of a system is generally more difficult to calculate than the total Shapley-Shubik power. To find the total Banzhaf power of a system, we must find the Banzhaf power of each of the voters in the system. But to find the total Shapley-Shubik power of a system, we must only determine the total number of arrangements of all of the voters in the system. The following question suggests how we might go about calculating this total number of arrangements.

Question 8.13.*

(a) How many different arrangements of two voters are possible?

(b) How many different arrangements of three voters are possible?

(c) How many different arrangements of four voters are possible? (Hint: You could write out each of these arrangements by hand, but it might be easier to just note that each arrangement of four voters can be formed by inserting the fourth voter into some already formed arrangement of the first three voters.)

(d) How many different arrangements of five voters are possible? (Hint: Use your answer to part (c) and the hint given there.)

(e) Do you see a pattern yet? Based on your answers to parts (a)–(d), how many different arrangements of six voters are possible? What about seven voters? Eight voters? n voters (where n just represents some arbitrary number of voters)?

The quantities you calculated in Question 8.13 are called *factorials*. For a whole number n, we write "n-factorial" as $n!$, and we define this quantity using the formula in the answer to part (e) of Question 8.13, which is given at the end of the chapter. (Note: You might want to jot down the formula for $n!$ in the margin here. We'll be using factorials quite a bit in our next round of calculations, so it will be convenient for you to have this formula at your fingertips.)

Now that we've seen how easy it is to find the total Shapley-Shubik power of a voting system, let's investigate the calculations involved in finding Shapley-Shubik indices for a more complicated example. We'll consider again the weighted voting system used by the 1965 Nassau County Board of Supervisors, which we first looked at in Question 8.7. Since there are six voters in the system (the six districts listed in Table 8.2), the total Shapley-Shubik power of the system is $6! = 720$. So all we need to do now is make a list of the 720 possible arrangements of the voters in the system and identify the pivotal voter in each one. Right?

Okay, maybe not. What we really need is a way to simplify this process, or at least break it down into smaller, more easily digestible chunks. The next question suggests one way to do just that.

Question 8.14.* Consider the weighted voting system from Question 8.7, with voters and weights as shown in Table 8.2, and a quota of 58.

(a) Suppose that in a particular arrangement of the six districts, the ordering of the three largest districts is Hempstead 2, followed in some later position by Oyster Bay, followed in some later position by Hempstead 1. Which of the six districts would be pivotal in such an arrangement?

(b) Does your answer to part (a) depend on where the three smallest districts are placed within the arrangement? Explain.

(c) Suppose that the ordering of the three smallest districts is North Hempstead, followed in some later position by Long Beach, followed in some later position by Glen Cove. How many different arrangements of all six districts would be consistent with this ordering *and* the ordering of the three largest districts specified in part (a)? (Hint: Each such arrangement is completely determined by which positions the largest three districts occupy. How many ways are there to choose these three positions?)

(d) Would your answer to part (c) be different if a different ordering of the three smallest districts were assumed?

(e) How many different orderings of the three smallest districts are possible?

(f) Using your answers to parts (b)–(e), determine the total number of arrangements of all six districts that have the three largest districts in the order specified in part (a).

(g) Would your calculations from parts (a)–(f) be different if a different ordering of the three largest districts were assumed?

(h) Using your answers to parts (a)–(g), determine the Shapley-Shubik index of each of the six districts in the system. Show all of your work, and clearly explain your reasoning.

Banzhaf Power in Psykozia

For the remainder of this chapter, we're going to apply what we've learned about the Banzhaf and Shapley-Shubik power indices to Psykozia's federal voting system, which we first considered in Question 7.14. Our main interest in looking at this system is due to its similarities to the United States' federal voting system. As such, the calculations we're about to do will give you a taste of what would be involved in determining the distribution of power within the United States' system.

Before we dive back into Psykozia, you should know that some of the calculations we'll need to do are quite complicated and involved. If you are a person who likes quick answers, the next several pages will be an exercise in patience and perseverance for you. With that said, know that what we're about to do isn't rocket science; it actually all boils down to multiplication and addition, which you definitely know how to do. And as we've seen before, the trick here will be to proceed in a careful and systematic way.

Now let's get started. Recall that in Psykozia's federal voting system, there are four different types of voters: senators, representatives, the president, and the vice president. We'll begin by calculating the Banzhaf index of one of the senators, whom we'll call S.

Question 8.15.* Recall that in Psykozia's federal voting system, there are four senators and five representatives.

(a) In how many different ways can a coalition consisting of S and two other senators be selected? Clearly explain your answer.

(b) In how many different ways can a coalition consisting of three representatives be selected? Clearly explain your answer.

(c) In how many different ways can a coalition consisting of S, two other senators, and three representatives be selected? Clearly explain your answer.

Hopefully, Question 8.15 warmed you up for the task of calculating the Banzhaf power of S. We'll do this in the next question.

Question 8.16.*

(a) Describe all of the different types of winning coalitions in which S would be critical. (Hint: These coalitions fall into 10 distinct categories, 3 that contain S and one other senator, and 7 that contain S and two other senators.)

(b) For each of the 10 different types of winning coalitions that you described in part (a), count the number of different ways in which that particular type of winning coalition could be formed. (Hint: Use the same kind of reasoning as you used in Question 8.15.)

(c) Using your answers to parts (a) and (b), determine the Banzhaf power of S.

The calculations required for Question 8.16 were somewhat tedious, but like many other things, they get easier once you've done a few. So let's continue by calculating the Banzhaf power of each of the other voters in the system.

Question 8.17.

(a) Using the same kind of reasoning you used in Question 8.16, determine the Banzhaf power of an individual representative in Psykozia's federal voting system. (Hint: You will need to consider nine different types of winning coalitions here.)

(b) Determine the Banzhaf power of the president and the vice president in Psykozia's federal voting system.

Question 8.18.*

(a) Using your answers to Questions 8.16 and 8.17, find the Banzhaf index of each of the voters in Psykozia's federal voting system. (Hint: Don't forget that there are four senators and five representatives.)

(b) According to the Banzhaf indices you calculated in part (a), what percentage of the total power in Psykozia's federal voting system is held by the Senate? What about the House of Representatives? The president? The vice president?

(c) Do you think your answers to parts (a) and (b) seem reasonable? Do you think anyone in the system has more power or less power than they deserve, or does the distribution of power seem about right? Explain.

That was a lot of work, but we were finally able to find the Banzhaf index of each of the voters in Psykozia's federal voting system. In just a moment, we'll also find the Shapley-Shubik index of each of these voters. But before we do so, let's take a few minutes to formalize some of the mathematical ideas we just used.

A Splash of Combinatorics

You may not have been aware of it, but when you answered the questions in the previous section, you were dipping your toe into an area of mathematics known as *combinatorics*. As you might guess from your answers to those questions, combinatorics focuses on problems that involve counting objects or combinations of objects, usually in a very precise and systematic way. Calculating Banzhaf indices often involves counting the number of different ways in which a certain set of objects can be chosen. For instance, in part (b) of Question 8.15, we needed to count the number of different ways in which we could choose three of the five representatives in Psykozia's federal voting system. With a little bit of thought, it's not too hard to see that there are exactly 10 ways to do this.

But what if the situation was more complicated? For instance, what if we needed to count the number of different ways to choose 51 out of the 100 senators in the United States' federal voting system? This is exactly the kind of question that the tools of combinatorics can help us answer. Let's begin with a bit of notation.

Definition 8.19. The number of different ways to choose k objects out of a collection of n objects is denoted by the symbol $\binom{n}{k}$, which is read "n choose k."

Question 8.20.* Find the value of each of the following quantities. (Note: Although you may already know a formula for $\binom{n}{k}$, you shouldn't need to use it for this question; just think about what each quantity represents.)

(a) $\binom{5}{0}$

(b) $\binom{5}{1}$

(c) $\binom{5}{2}$

(d) $\binom{5}{3}$

(e) $\binom{5}{4}$

(f) $\binom{5}{5}$

(g) $\binom{5}{1} + \binom{5}{2}$

(h) $\binom{6}{2}$

If you look carefully at your answers to Question 8.20, you'll see evidence of a number of useful properties. For example, it must be the case that $\binom{5}{0} = 1$, since there is exactly one way to choose zero objects out of a collection of five objects. (The one way is to not choose any of the five objects.) But the same reasoning would also apply if we were considering a collection of six or eight or a hundred objects. Thus, we have the following result.

Theorem 8.21. *For any value of* n, $\binom{n}{0} = 1$.

As simple as it is, the previous theorem is one of many useful facts that will help us be able to find $\binom{n}{k}$ for any values of n and k we desire. In the next two questions, we'll identify some of the other properties that will also be helpful in this regard.

Question 8.22.* What is the relationship between $\binom{n}{k}$ and $\binom{n}{n-k}$? Give a convincing argument to explain why this relationship will hold for all possible values of n and k. (Hint: It might be helpful to look back at your answers to Question 8.20.)

Question 8.23.* Fill in the blanks to make each of the following statements true. Then give a convincing argument to explain why each statement is true. (Hint: You may want to look at some examples first, especially for part (c).)

(a) For any value of n, $\binom{n}{1} = \underline{\qquad\qquad}$.

(b) For any value of n, $\binom{n}{n} = \underline{\qquad\qquad}$.

(c) For any values of n and k, $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{\underline{\qquad\qquad}}$.

Now let's put all of these facts together. In doing so, we'll catch a glimpse of a famed mathematical tool known as Pascal's triangle, named after 17th-century mathematician Blaise Pascal. Pascal's triangle contains the various values of $\binom{n}{k}$ in a triangular array, the top four rows of which are shown in Figure 8.1. The triangle continues on forever, following the same pattern, with each row containing one more entry than the row above it.

$$
\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \\
 & & & \vdots & & &
\end{array}
$$

FIGURE 8.1. Pascal's triangle

If we label the rows of Pascal's triangle starting at the top with zero, and also label the entries in each row starting at the left with zero, then the kth entry in the nth row is $\binom{n}{k}$. The top four rows with actual numerical values for $\binom{n}{k}$ are shown in Figure 8.2.

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 2 & & 1 & \\
1 & & 3 & & 3 & & 1 \\
 & & & \vdots & & &
\end{array}
$$

FIGURE 8.2. Pascal's triangle

Question 8.24.

(a) What number appears at the beginning and at the end of each row of Pascal's triangle? Which property of $\binom{n}{k}$ lets you conclude this?

(b) Explain how the numbers in any row of Pascal's triangle can always be found from the numbers in the row above it. (Hint: You might want to look back at your answer to part (c) of Question 8.23.)

Question 8.25.*

(a) Using your answers to Question 8.24, write the next four rows of Pascal's triangle (after the last row shown in Figure 8.2).

(b) Use your answer to part (a) to find $\binom{7}{3}$.

(c) Suppose you needed to find $\binom{9}{6}$. How could you do it?

Shapley-Shubik Power in Psykozia

Now that we know some combinatorics, we're finally ready to calculate the Shapley-Shubik indices of the voters in Psykozia's federal voting system. Just like we did when we considered Banzhaf power in Psykozia, we'll begin here by calculating the Shapley-Shubik index of one of the four senators, whom we'll again call S.

Question 8.26.*

(a) Suppose all of the voters in Psykozia's federal voting system are arranged in some order so that S is preceded by exactly two other senators and three representatives. Is S pivotal in this type of arrangement?

(b) Suppose all of the voters in Psykozia's federal voting system are arranged in some order so that S is preceded by exactly two other senators and four representatives. Is S pivotal in this type of arrangement?

(c) In how many different ways can the six voters preceding S in part (b) be selected?

(d) Once the six voters preceding S in part (b) are selected, in how many different ways can these six voters be arranged in some order?

(e) Once the six voters preceding S in part (b) are selected and arranged, how many voters will be left to follow S in an arrangement of all of the voters in the system?

(f) In how many different ways can the voters following S in part (e) be arranged in some order?

(g) Based on your answers to parts (c)–(f), in how many different ways can all of the voters in Psykozia's federal voting system be arranged so that S is preceded by exactly two other senators and four representatives?

Question 8.27.*

(a) Describe all of the different types of arrangements of the voters in Psykozia's federal voting system in which S would be pivotal. (Hint: These arrangements fall into ten distinct categories, one of which is identified in part (b) of Question 8.26.)

(b) Calculate the number of different ways in which each type of arrangement from part (a) could be formed. (Hint: Use the same kind of reasoning as you used in Question 8.26.)

(c) Using your answers to parts (a) and (b), determine the Shapley-Shubik index of S.

Question 8.28.* Using the same kind of reasoning as you used in Question 8.27, determine the Shapley-Shubik index of an individual representative in Psykozia's federal voting system. Then do the same for the president and the vice president.

Question 8.29.

(a) Does anything about the Shapley-Shubik indices of the voters in Psykozia's federal voting system strike you as being strange or unusual? Explain.

(b) Compare the Banzhaf and Shapley-Shubik indices of the voters in Psykozia's federal voting system. Do you think these two indices will always give such similar results? If so, explain why. Otherwise, give an example of a system for which the results of the two indices would be significantly different.

Question 8.30. Write a page or two comparing Banzhaf's ideas on how power is distributed in yes/no voting systems to Shapley and Shubik's. Include in your summary answers to at least the following questions:

- Which index do you think more accurately represents the distribution of power within a system: the Banzhaf index or the Shapley-Shubik index? Clearly explain your reasoning.
- Do you think that the index you specified above will be better all of the time, most of the time, or just some of the time? Explain.
- Which do you think play a more important role in yes/no voting systems, critical voters or pivotal voters? Explain.

Questions for Further Study

Question 8.31. Using the idea of a critical voter, write new definitions for the terms *dictator*, *veto power*, and *dummy* (originally defined in Definition 7.10). Then do the same thing using the idea of a pivotal voter.

Question 8.32.

(a) What must the sum of the Banzhaf indices of all of the voters in a yes/no voting system be? Give a convincing argument to justify your answer.

(b) Would your answer to part (a) be different for Shapley-Shubik indices instead of Banzhaf indices? Why or why not?

Question 8.33. Consider again the weighted voting system from Warmup 8.1. Calculate the Banzhaf and Shapley-Shubik indices of each of the voters, but this time assume that the quota for the system is 101. Then repeat the calculations for a quota of 105.

Question 8.34.

(a) Find the Banzhaf index of each of the voters in the weighted voting system $[65 : 30, 28, 22, 15, 13]$.

(b) Note that in the weighted voting system used by the Nassau County Board of Supervisors in 1994 (see Question 8.8), if the Long Beach and Glen Cove districts agreed to always vote together, then the voting system used by the board would be equivalent to the system in part (a). With this in mind, use your answers to part (a) and Question 8.8 to answer the following questions:

- If the Long Beach and Glen Cove districts agreed to always vote together, would they have more, less, or the same amount of combined power as they had before the agreement?
- Would the Banzhaf indices of any of the other districts be affected by this agreement?
- Does anything about your answers to parts (a) and (b) strike you as being strange or unusual? Explain.

Question 8.35. Calculate the Shapley-Shubik index of each of the six districts in the weighted voting system used by the Nassau County Board of Supervisors in 1994. (See Question 8.8.) How do the Shapley-Shubik indices compare to the Banzhaf indices that you calculated for the same system in part (a) of Question 8.8? Which index do you think gives a better representation of how the power is truly distributed in the system? Clearly explain your answer.

Question 8.36. Write a short biography of John F. Banzhaf III, including the academic positions he has held and some information about his most notable court cases.

Question 8.37. Write short biographies of Lloyd Shapley and Martin Shubik, including where they met, where they were employed when they came up with the idea for their index, and any other notable contributions they have made both inside and outside of voting theory.

Question 8.38. The quantities we denoted by $\binom{n}{k}$ are often called *binomial coefficients*. Research the meaning of this name, and write a summary of your findings. Include in your summary a description of at least one mathematical application of binomial coefficients outside of voting theory.

Question 8.39. In combinatorics, the quantity $\binom{n}{k}$ is often defined by the following formula:

$$\binom{n}{k} = \frac{n!}{k! \times (n-k)!}.$$

Explain why this definition of $\binom{n}{k}$ is completely consistent with the definition we used in this chapter.

Question 8.40. It is a well-known fact in combinatorics that for any value of n,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n.$$

Without doing any calculations, explain why this equation is true. (Hint: Explain how the two sides of the equation can be viewed as two different ways of counting the same thing.)

Question 8.41. Recall from Question 7.13 that the voting system used to make decisions on motions in the U.N. Security Council can be viewed as the weighted system $[39 : 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$. Find the Banzhaf and Shapley-Shubik indices of each of the voters in this system.

Question 8.42. In Question 7.27, we considered the voting system used to amend the Constitution of Canada. Find the Banzhaf index of each of the voters in this system (the ten Canadian provinces), and comment on anything about the results that strikes you as being strange or unusual.

Question 8.43. Suppose that in the country of Psykozia, the states of Ignorance and Bliss decide to merge to form a new state, Enlightenment. (So, in this case, ignorance really is bliss!) What corresponding changes do you think should be made to Psykozia's federal voting system? With these changes, would the people of Enlightenment have more, less, or the same amount of combined power as they had before the merger?

Question 8.44. Research the voting system used by the original European Economic Community (EEC), which was established in 1958 as the predecessor to the European Union. Find the Banzhaf and Shapley-Shubik indices of each of the voters in the voting system used by the original EEC. Explain any discrepancies between the two indices, and comment on anything that strikes you as being strange or unusual about how power was distributed within the EEC.

Question 8.45. Find a book or article that gives the Banzhaf or Shapley-Shubik indices of each of the voters in the United States' federal voting system, and write a summary of your findings. Include in your summary answers to at least the following questions: Did the book or article mention how the calculations were done? Do you think the indices accurately represent how the power is truly distributed in the system? Does anything about the indices strike you as being strange or unusual? (Note: There are a total of 537 voters in the system—100 senators, 435 representatives, the president, and the vice president.)

Question 8.46. In addition to the Banzhaf and Shapley-Shubik power indices, there are a number of other methods that can be used to measure how power is distributed within a yes/no voting system. Two of the more common measures are the Deegan-Packel index and the Johnston index. Research both of these indices, and write a detailed summary of your findings. Include in your summary a comparison of the two indices to each other and to the Banzhaf and Shapley-Shubik indices. Also calculate the values of the Deegan-Packel and Johnston indices for each of the six districts in the weighted voting system from Question 8.8.

Question 8.47. Calculate the Banzhaf and Shapley-Shubik indices of each of the voters in the system from Question 7.39.

Question 8.48. Calculate the Banzhaf and Shapley-Shubik indices of each of the voters in the system from Question 7.40.

Question 8.49. Consider a weighted voting system with three voters. For each of the following distributions of power, either find weights and a quota for which the Shapley-Shubik index would yield the given distribution, or explain why it is impossible to do so.

(a) $1, 0, 0$

(b) $\frac{5}{6}, \frac{1}{6}, 0$

(c) $\frac{4}{6}, \frac{2}{6}, 0$

(d) $\frac{4}{6}, \frac{1}{6}, \frac{1}{6}$

(e) $\frac{3}{6}, \frac{3}{6}, 0$

(f) $\frac{3}{6}, \frac{2}{6}, \frac{1}{6}$

(g) $\frac{2}{6}, \frac{2}{6}, \frac{2}{6}$

Question 8.50. Using the Shapley-Shubik index to measure power, find a weighted voting system with four voters (call them A, B, C, and D) for which:

- A is twice as powerful as B;
- B is twice as powerful as C; and
- C and D have the same amount of power.

Question 8.51. Show that the Banzhaf and Shapley-Shubik indices are always identical when there are only two voters.

Answers to Starred Questions

8.3. (a) The winning coalitions are {Doug, Nicholas}, {Doug, Elisabeth}, and {Doug, Nicholas, Elisabeth}.

(b) Doug is critical in all three winning coalitions. Nicholas and Elisabeth are each critical in only one winning coalition.

(c) The Banzhaf power of Doug is 3, and the Banzhaf power of both Nicholas and Elisabeth is 1. The total Banzhaf power of the system is 5.

(d) The Banzhaf index of Doug is 3/5, and the Banzhaf index of both Nicholas and Elisabeth is 1/5.

8.7. (a) This question is much easier than it seems. Take two of the three largest districts along with zero of the three smallest districts, and see what the weights of the resulting coalitions would be. Then take one of the three largest districts along with all three of the smallest districts, and see what the weights of the resulting coalitions would be.

(b) There are a total of 32 winning coalitions for the system. Of these, 3 have two members, 10 have three members, 12 have four members, 6 have five members, and 1 has six members.

(c) There are a total of 48 critical voters in all of the winning coalitions for the system.

(d) According to your answers to parts (a) and (c), before you even start part (d) you should know what the Banzhaf powers and indices will end up being for each of the six districts.

8.8. (a) There are a total of 23 winning coalitions for the system. Of these, 5 have three members, 11 have four members, 6 have five members, and 1 has six members. There are a total of 52 critical voters in all of the winning coalitions for the system.

(b) In the 1994 system, the percentage of power held by Hempstead 2 was exactly 25%, and the percentage of power held by Glen Cove was 1.92%.

8.10. (a) There are six different arrangements of Doug, Nicholas, and Elisabeth. Abbreviating using just the first letter of each name, these six arrangements are DNE, DEN, NDE, NED, EDN, and END.

(b) Doug is pivotal in four arrangements. Nicholas and Elisabeth are each pivotal in only one arrangement.

(c) Doug has a Shapley-Shubik index of 4/6, and Nicholas and Elisabeth each have a Shapley-Shubik index of 1/6.

8.13. (a) For two voters, there are only 2 possible arrangements.

(b) For three voters, there are 6 possible arrangements.

(c) For four voters, there are 24 possible arrangements. These arrangements can be found by inserting the fourth voter into each of 4 different places in each of the 6 arrangements from part (b).

(d) For five voters, there are $5 \times 24 = 120$ possible arrangements.

(e) For n voters, there are $n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$ possible arrangements.

8.14. (a) Oyster Bay would be pivotal.

(b) Oyster Bay would be pivotal regardless of where the three smallest districts were placed within the arrangement.

(c) There are 20 different arrangements that would be consistent with both orderings. If you abbreviate the names of the districts, you should be able to write them all down.

(d) The answer to part (c) would be the same, even if a different ordering of the three smallest districts were assumed.

(e) There are six different orderings of the three smallest districts.

(f) For each ordering of the three smallest districts, there are 20 arrangements of all six districts that have the three largest districts arranged in the order specified in part (a). Since there are six possible orderings of the three smallest districts, there must be $6 \times 20 = 120$ total arrangements of all six districts that adhere to the ordering of the three largest districts specified in part (a).

(g) If a different ordering of the three largest districts were assumed, then a different district might be pivotal. Other than this, all of the calculations from parts (a)–(f) would remain the same.

(h) The Shapley-Shubik indices for the districts should be the same as the Banzhaf indices you found in part (d) of Question 8.7.

8.15. (a) There are 3 different ways to select a coalition consisting of S and two other senators.

(b) There are 10 different ways to select a coalition consisting of three representatives.

(c) For each of the 3 different ways to select S and two other senators, there are 10 different ways to select three representatives. Thus there are $3 \times 10 = 30$ different ways to select a coalition consisting of S, two other senators, and three representatives.

8.16. (a) The following are a few of the 10 different types of winning coalitions in which S would be critical:

- S, one other senator, the president, the vice president, and four representatives;
- S, two other senators, and four representatives;
- S, two other senators, the president, and three representatives;
- S, two other senators, the vice president, and four representatives.

(b) Consider the first type of coalition listed in the answer to part (a). To form this type of coalition, we must choose one of the three senators besides S (3 possible choices), and four of the five representatives (5 possible combinations). Thus, there are $3 \times 5 = 15$ ways in which this type of coalition could be formed. The same kind of reasoning will work for the other 9 types of winning coalitions in which S would be critical.

(c) You should find that the Banzhaf power of S is 132.

8.18. The Banzhaf indices of the voters in the system are:

- $\dfrac{132}{1500}$ for each of the senators;
- $\dfrac{136}{1500}$ for each of the representatives;
- $\dfrac{196}{1500}$ for the president; and
- $\dfrac{96}{1500}$ for the vice president.

8.20. (a) 1

(b) 5

(c) 10

(d) 10

(e) 5

(f) 1

(g) 15

(h) 15

8.22. If you're having trouble explaining your answer, consider this: If you choose k objects out of a collection of n objects, how many of the n objects will *not* be chosen?

8.23. (c) For any values of n and k,

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

To see why this is true, suppose you chose k widgets out of a collection of $n+1$ widgets, one of which was red and n blue. If the red widget was among the k that you chose, then how many blue widgets would you have chosen? If the red widget was not among the k that you chose, then how many blue widgets would you have chosen? Now try to extend this reasoning to formulate a general argument.

8.25. (a)

$$\begin{array}{ccccccccccccccc}
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}$$

(b) $\binom{7}{3} = 35$

8.26. (a) S is not pivotal in this type of arrangement.

(b) S is pivotal in this type of arrangement.

(c) There are $\binom{3}{2} \times \binom{5}{4} = 15$ different ways in which these six voters can be selected.

(d) There are $6! = 720$ different ways in which the six selected voters can be arranged.

(e) There will be four voters left over: a senator, a representative, the president, and the vice president.

(f) There are $4! = 24$ different ways in which the four voters that are left can be arranged.

(g) There are $15 \times 720 \times 24 = 259{,}200$ different ways in which all of the voters in the system can be arranged in some order so that S is preceded by exactly two other senators and four representatives.

8.27. The Shapley-Shubik index of S is approximately .087, or 8.7%.

8.28. The Shapley-Shubik index of an individual representative is approximately .091, or 9.1%. The Shapley-Shubik index of the president is approximately .136, or 13.6%, and the Shapley-Shubik index of the vice president is approximately .061, or 6.1%.