Pigeonhole: Old and preliminary results

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June 13, 2024

RaTLoCC24
Pisa, Italy

Reverse mathematics

Reverse mathematics examines the logical strength of theorems via proofs in a hierarchy of subsystems of second order arithmetic.

Language:
Variables for numbers (type 0) and sets of numbers (type 1)

Base axiom system: RCA_0
Axioms: Basic arithmetic
$\mathsf{I}\Sigma^0_1$: Induction for $\exists n$ formulas
Recursive comprehension
(sets with computable characteristic functions)

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Induction and bounding schemes:

$$\mathrm{I}\Sigma^0_1 < \mathrm{B}\Pi^0_1 < \mathrm{I}\Sigma^0_2 < \mathrm{B}\Pi^0_2 < \dots$$

RCA_0 cannot prove $\mathrm{B}\Pi^0_1$. For details, see Chapter 6 of Dzhafarov and Mummert's text [1].

An old pigeonhole theorem

Theorem: (RCA_0) The following are equivalent:

(1) $\mathrm{B}\Pi^0_1$: If $\theta(x, y, z)$ is a quantifier free formula, and $(\forall x < a)(\exists y)(\forall z)\theta(x, y, z)$ then

$$(\exists b)(\forall x < a)(\exists y < b)(\forall z)\theta(x, y, z)$$

(2) RT1: If $f : \mathbb{N} \to m$ then for some $j < m$, the set $\{n \mid f(n) = j\}$ is infinite.

This theorem is included in Hirst's thesis, but it's easier to find the proof in Dzhafarov and Mummert's text [1].

A preliminary result

Theorem [H,K,M,R[1]] (RCA_0) The following are equivalent:

(1) PHB: If $f : \mathbb{N} \to m$ then f has a pigeonhole basis, that is, a set B of exactly those colors that appear infinitely often in the range of f.

(2) $\mathsf{I}\Sigma^0_2$: the induction scheme for $\exists x \forall y$ formulas.

[1] Authors include the members of the 2024 Noncomputability course at Appalachian State University: Silva Keohulian, Brody Miller, and Jessica Ross.

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This shows that PHB is strictly stronger than RT1.

Blueprint for the proof: Prove that PHB is equivalent to bounded Π^0_2 comprehension, and invoke Ex. II.3.13 of Simpson [5].

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Ideas from the proof

- Bounded Π^0_2 comprehension $\to$ PHB

Suppose $f : \mathbb{N} \to m$. The pigeonhole basis is

$$\{j < m \mid \forall s \exists n(n > s \wedge f(n) = j)\}$$

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- PHB $\to$ bounded Π^0_2 comprehension

A concrete example: We want to find

$$S = \{j < 2 \mid \forall s \exists n\, \theta(j, s, n)\}.$$

Suppose $\theta(0, 0, 1)$, $\theta(0, 1, 3)$, $\theta(0, 2, 10)$, and so on.
Suppose $\theta(1, 0, 0)$, but $\neg\exists n\, \theta(1, 1, n)$. (Want $S = \{0\}$.)

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Computation of the function f

n	0	1	2	3	4	5	6
f for $j = 0$							
f for $j = 1$							

Markers

$j = 0$	0		
$j = 1$	0		

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Computation of the function f

n	0	1	2	3	4	5	6
f for $j = 0$	3						
f for $j = 1$							

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n	0	1	2	3	4	5	6
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Computation of the function f

n	0	1	2	3	4	5	6
f for $j = 0$	3		0		0		
f for $j = 1$							

Markers

$j = 0$	0	1	2
$j = 1$	0		

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Computation of the function f

n	0	1	2	3	4	5	6
f for $j = 0$	3		0		0		3
f for $j = 1$							

Markers

$j = 0$	0	1	2
$j = 1$	0		

Ideas from the proof

• Bounded Π^0_2 comprehension $\rightarrow$ PHB

Suppose $f : \mathbb{N} \rightarrow m$. The pigeonhole basis is

$$\{\underbrace{j < m}_{\text{bound}} \mid \underbrace{\forall s \exists n (n > s \wedge f(n) = j)}_{\Pi^0_2 \text{ formula}}\}$$

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Computation of the function f

n	0	1	2	3	4	5	6
f for $j = 0$	3		0		0		3
f for $j = 1$		1					

Markers

$j = 0$	0	1	2
$j = 1$	0	1	

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Computation of the function f

n	0	1	2	3	4	5	6
f for $j = 0$	3		0		0		3
f for $j = 1$		1		3		3	

Markers

$j = 0$	0	1	2
$j = 1$	0	1	X

Ideas from the proof

•PHB $\rightarrow$ bounded Π^0_2 comprehension

A concrete example: We want to find

$$S = \{j < 2 \mid \forall s \exists n \, \theta(j, s, n)\}.$$

Suppose $\theta(0, 0, 1)$, $\theta(0, 1, 3)$, $\theta(0, 2, 10)$, and so on.
Suppose $\theta(1, 0, 0)$, but $\neg \exists n \, \theta(1, 1, n)$. (Want $S = \{0\}$.)

Computation of the function f

n	0	1	2	3	4	5	6
f for $j = 0$	3		0		0		3
f for $j = 1$		1		3		3	

Markers

$j = 0$	0	1	2
$j = 1$	0	1	X

Values of f are 3, 1, 0, 3, 0, 3, 3, . . . (Infinitely many 0s and 3s.)

PHB(f)=$\{0, 3\}$. Delete the 3 to get $S = \{0\}$.

Enumerated matroids

Theorem [H,Mummert] (RCA_0) The following are equivalent:

(1) EMB: Every finite dimensional e-matroid has a basis.

(2) $\mathsf{I}\Sigma^0_2$.

An e-matroid (M, e) consists of a set (like vectors) and a function $e : \mathbb{N} \to M^{<\mathbb{N}}$ that enumerates all the finite dependent sets. Requirements:

(1) $\emptyset$ is independent.

(2) Finite supersets of dependent sets are dependent.

(3) Exchange principle: If X and Y are independent and $|X| < |Y|$, then for some $y \in Y$, $X \cup \{y\}$ is independent.

A *basis* is a maximal independent set.

(e, M) is finite dimensional if e enumerates all the sets bigger than size b for some b.

A direct proof

Theorem (RCA_0) EMB $\to$ PHB.

Sketch: Suppose $f : \mathbb{N} \to m$ is a PHB instance.

Preprocessing: Turn f into an EMB instance.

Let $f'(n)$ be the identity if $n < m$ and $f(n)$ if $n \geqslant m$. Note that PHB(f')=PHB(f). List all finite subsets of $\mathbb{N}$ ($S_0, S_1, \ldots$) with each set repeated infinitely many times. The set $S_n = \{x_0, x_1, \ldots, x_k\}$ is *dependent* if there is an $x_i < m \leqslant n$ such that $f'(x_i) = f'(m)$. Define $e(n) = S_n$ if S_n is dependent, and $e(n) = [0, m]$ otherwise.

Apply EMB to find a basis for $(e, \mathbb{N})$. Call it B'.

Postprocessing: Turn B' into a PHB for f.

$x \in B'$ if and only if x is the largest value colored $f'(x)$. So $B = [0, m) - \{f'(x) \mid x \in B'\}$ is the pigeonhole basis for f' (and also for f).

Weihrauch reductions

Problems: Input a set and output a set (or number)

PHB is a problem.

An instance of PHB is a function $f : \mathbb{N} \to m$.

Realizer: A function mapping instances to solutions.

We say P is Weihrauch reducible to Q (and write $P \leqslant_W Q$) if there are (partial) computable procedures Φ (for preprocessing) and Ψ (for postprocessing) such that if p is an instance of P, then

- $\Phi(p)$ is an instance of Q and
- for any solution s of $\Phi(p)$, $\Psi(s, p)$ is a solution of p.

That is, if R_Q is any realizer of Q, then $\Psi(R_Q(\Phi(p)), p)$ is a realizer for P.

Weihrauch analysis of PHB

Thm [H,K,M,R] PHB$\leqslant_W$EMB.

Proof: Use the sketch for EMB$\to$PHB.

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Thm [H,K,M,R] PHB$\leqslant_W$EMB.

Proof: Use the sketch for EMB$\to$PHB.

Another example:

The problem LPO (Limited Principle of Omniscience)
Input: $f : \mathbb{N} \to 2$ Output: 0 if 0 is in the range of f, 1 if not.

Thm [H,K,M,R] LPO$\leqslant_W$PHB.

Sketch: If f is (for example):

n	0	1	2	3	4
f(n)	1	1	0	1	0

then $\Phi(f)$ is

n	0	1	2	3	4
Φ(f)(n)	1	1	0	0	0

. For an arbitrary $g : \mathbb{N} \to 2$, if g has a zero, then PHB$(\Phi(g)) = \{0\}$, and if g has no zeros then PHB$(\Phi(g)) = \{1\}$, matching LPO(g).

Comparison of results

- Reverse mathematics:

RCA_0 proves PHB$\leftrightarrow$EMB.

RCA_0 proves LPO (by classical logic), so LPO$\not\rightarrow$PHB.

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RCA_0 proves LPO (by classical logic), so LPO$\not\to$PHB.

- Weihrauch reducibility

LPO$<_W$PHB$<_W$EMB

The proofs that PHB$\not\leq_W$LPO and EMB$\not\leq_W$PHB are somewhat technical proofs by contradiction.

- In both settings, LPO is strictly weaker than PHB.

Higher order reverse mathematics

Kohlenbach [4] introduced an extension of reverse mathematics to higher types. The language is richer, including functions from sets to numbers (type 2), for example.

If P is a Weihrauch problem, we can write (P) for the principle that asserts the existence of a realizer for P.

The relationship between $P \leqslant_{\mathrm{W}} Q$ and the theorem $(Q) \to (P)$ is often surprising.

A higher order result

Theorem [H,K,M,R] (RCA_0^ω) The following are equivalent:

(1) (PHB): There is a function R_{PHB} such that if $f : \mathbb{N} \to m$, then $R_{PHB}(f)$ is the pigeonhole basis for f.

(2) (LPO): There is a function R_{LPO} such that if $f : \mathbb{N} \to 2$, then $R_{LPO}(f)$ is 0 if 0 is in the range of f and 1 otherwise.

Comments on the proof:

(1) implies (2) is a formalization of LPO$\leqslant_W$PHB

(2) implies (1) depends on the fact that RCA_0^ω allows sequential applications of R_{LPO}. We could formalize a proof of PHB$\leqslant_W \widehat{\text{LPO}}$.

(LPO) is the same as Kohlenbach's $(\exists^2)$, and related to Kleene's E2.

Immediate questions

Does RCA_0^ω prove (LPO)$\leftrightarrow$(EMB)?

What about results for fixed (rather than bounded) dimension?

What about similar basis results for RT2 or RT3 or HT?
(Lists of infinite monochromatic sets in each possible color?)

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